# Chapter 1 Module

## § 1.1 Definition and Propeties of Module

Assume $R$ to be a unital commutative ring with unit $1_R$ (or simply 1). We also assume $0_R \neq 1_R$ to avoid some bad propety of the ring $R$.

<u>Def 1.1</u> Let $(R,+,\cdot)$ be a ring, $(M,+)$ be an Abelian group, if there is a map $R\times M \to M$, $(a,m)\mapsto a\cdot m$ such that:

(1) $a(x+y) = ax + ay$

(2) $(a+b)x = ax + bx$

(3) $(ab)x = a(bx)$

(4) $1x = x$

then $M$ is called a left $R$ module.

<u>Remark</u>. The right module can be defined similaly. When $R$ is a commutative ring, a left $R$-module is also a right $R$-module. (Def. 1.2)

<u>Example</u> 1.1 Vector spaces are modules over fields.

<u>Example</u> 1.2 Any Abelian group $M$ is a $\mathbb{Z}$-module with action given by

$$m\cdot x := \underbrace{x+\cdots+x}_{m},\ 0_{\mathbb{Z}}\cdot x := 0_M,\ (-1)\cdot x = -x.$$

<u>Example</u> 1.3 Let $V$ be a vector space over field $\mathbb{F}$ and $T$ be a linear operator $T: V\to V$. Consider polynomial ring $\mathbb{F}[\lambda]$, we have $\mathbb{F}[\lambda]\cong\mathbb{F}[T]$, via the action of $T$ on $V$, we obtain a $\mathbb{F}[\lambda]$ module structure over $V$,

$$\mathbb{F}[\lambda]\times V \to V,$$

$$(p(\lambda),x)\mapsto p(T)\cdot x.$$

<u>Example</u> 1.4 Ring $R$ is a module over itself. If $S$ is a subring of $R$, then $R$ is a $S$-module, but $S$ is not necessarilly a $R$-module. When $S$ is a ideal of $R$ (all ideals are subrings), $S$ is an $R$-module. Notice $R$ is a subring of $R[x_1,\cdots,x_n]$, thus $R[x_1,\cdots,x_n]$ is an $R$-module. Similarly, the ring of formal power series $R[[x]]$ is an $R$-module.

<u>Example</u> 1.5 Let $R,S$ be ring and $\psi: R\to S$ be a ring homomorphism, $M$ be a $S$-module, then $M$ is an $R$-module with action $a\cdot x := \psi(a)\cdot x$.

<u>Claim</u> Let $M$ be an $R$-module, we have:

(1) $a\cdot 0 = 0$ (1') $0_R x = 0_M$

(2) $a\cdot(-x) = -(ax)$ (2') $(-a)\cdot x = -(a\cdot x)$

(3) $a\cdot \sum_i x_i = \sum_i a\cdot x_i$ (3') $(\sum_i a_i)\cdot x = \sum_i (a_i\cdot x)$

Proof. (1) $a\cdot x = a\cdot(x+0) = a\cdot x + a\cdot 0 \Rightarrow a\cdot 0 = 0$.

(1') $a\cdot x = (a+0)\cdot x = a\cdot x + 0\cdot x \Rightarrow 0\cdot x = 0$

(2) $a\cdot(-x) + a\cdot x = a\cdot 0 = 0 \Rightarrow a\cdot(-x) = -a\cdot x$

(2') $(-a)\cdot x + a\cdot x = [(-a)+a]\cdot x = 0 \Rightarrow (-a)\cdot x = -a\cdot x$

(3) and (3') are obvious.

Mainly use the cancellation property of group.

Example 1.6 Zero module $M = \{0\}$.

Def. 1.3 Let $M$ be an $R$-module and $N \neq \phi$ a subset of $M$. $N$ is called a submodule of $M$ if

(1) $N$ is a subgroup of $M$;

(2) $ax \in N$, $\forall a \in R$, $\forall x \in N$.

Prop 1.1 The non-empty subset $N \subseteq M$ is a submodule iff

(1) $\forall y_1, y_2 \in N$, $y_1 + y_2 \in N$

(2) $\forall a \in R$, $\forall x \in N$, $a\cdot x \in N$

Proof. "$\Rightarrow$" is obvious

"$\Leftarrow$": Choose $a=0$ in (2), and $y \in N$, based on claim above, we obtain $0\cdot y = 0 \in N$.

Similarly, choose $a=-1$ and $y \in N$, $(-1)\cdot y = -y \in N$. Thus $N$ is a subgroup.

Example 1.7 Let $M$ be a $\mathbb{Z}$-module, $N$ is a submodule iff $N$ is an additive subgroup.

Example 1.8 Let $V$ be a vector space over $\mathbb{F}$, $W$ is a submodule iff $W$ is a subspace.

Example 1.9 Let $V$ be a vector space over field $\mathbb{F}$, $T$ be a linear map $T: V \to V$. Regard $V$ as a $\mathbb{F}[T]$ module, a submodule of $V$ is an invariant subspace of $T$.

Proof. "$\Rightarrow$" $N$ is a submodule of $V$, show $N$ is invariant subspace.

$N$ is a subgroup, since $\mathbb{F} \subseteq \mathbb{F}[T]$, $N$ is closed under the action of $\mathbb{F}$, thus $N$ is a subspace.

Since $T \in \mathbb{F}[T]$, $N$ should be closed under the action of $T$.

"$\Leftarrow$" $N$ is an invariant subspace of $V$, show that $N$ is a submodule.

$N$ is a subgroup of $V$. $N$ is invariant under the action of $T$, thus it is also invariant under the action of $T^m$ $(m=1,2,3,\cdots)$, this implies $N$ is invariant under the action of $p(T) \in \mathbb{F}[T]$.

**Example 1.10** Regard ring $R$ as $R$-module, $N$ is a submodule iff $N$ is an ideal.

This is clear from definition.

**Example 1.11** Let $\{N_i \mid i \in I\}$ be a family of submodules, then $\bigcap_{i\in I} N_i$ is a submodule.

**Example 1.12** Zero submodule $\{0\}$.

**Example 1.13** For ring $R$ and $R$ module $M$, let $x \in M$, we define annihilator of $x$ as

$$\mathrm{Ann}_R(x) := \{a \in R \mid a\cdot x = 0\}.$$

(1) $\mathrm{Ann}_R(x)$ is an ideal of $R$

Proof. Step 1. Show $\mathrm{Ann}_R(x)$ is a subgroup.

Suppose $a, b \in \mathrm{Ann}_R(x)$, $a\cdot x = b\cdot x = 0$, then $(a-b)\cdot x = 0$. This implies $a-b \in \mathrm{Ann}_R(x)$, $\mathrm{Ann}_R(x)$ is a subgroup.

Step 2. Show $R\cdot \mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R(x)$.

$\forall r \in R$ and $a \in \mathrm{Ann}_R(x)$, $(r\cdot a)\cdot x = r\cdot(a\cdot x) = r\cdot 0 = 0$, $\Rightarrow r\cdot a \in \mathrm{Ann}_R(x)$.

(2) If $\mathrm{Ann}_R(x) \neq 0$, $x$ is called a torsion element.

(3) If $R$ is an integral domain (commutative ring such that, $\forall a, b \in R$, $a\cdot b = 0$ implies $a = 0$ or $b = 0$), the set of all torsion elements $T(M)$ is a submodule, called torsion submodule. If $M = T(M)$, $M$ is called a torsion module. $M$ is called torsion free if $T(M) = 0$.

Proof. Step 1. Show $T(M)$ is a subgroup.

If $x \in T(M)$, $\exists a \in R$ $a \neq 0$ s.t. $a\cdot x = 0$, this implies that $a\cdot(-x) = -a\cdot x = 0 \Rightarrow -x \in T(M)$. $0 \in T(M)$ is obvious.

If $x, y \in T(M)$, there exist nonzero $a, b \in R$ such that $a\cdot x = b\cdot y = 0$. Set $r = a\cdot b$, we have $r\cdot x = r\cdot y = 0$, where $r \neq 0$ since $a, b \neq 0$. (Property of integral domain)

Step 2. Show $\forall a \in R$, $\forall x \in T(M)$, $a\cdot x \in T(M)$.

Since $x \in T(M)$, $\exists b \in R$, $b \neq 0$ s.t. $b\cdot x = 0$. This implies that $b\cdot(ax) = a\cdot(bx) = a\cdot 0 = 0 \Rightarrow a\cdot x \in T(M)$.

(4) If $G$ is finite Abelian group, when regarded as $\mathbb{Z}$-module, we have $T(G) = G$.

Proof. Since all elements in finite group are of finite order.

(5) For vector space $V$, when regarded as $\mathbb{F}[K]$-module for some linear map $K: V \to V$, we have

$$T(V) = V.$$

Proof. Suppose $\dim V = d$, for $v \in V$, if $v \in \mathrm{Ker}\, K$, $Tv = 0 \Rightarrow v \in T(V)$.

If $v \notin \operatorname{Ker} k$, $v, Tv, \cdots, T^d v, T^{d+1} v$ must be linear dependent, there exist nonzero $(\alpha_0, \alpha_1, \cdots, \alpha_{d+1})$ such that $\Sigma_i \alpha_i T^i v = 0$, meaning $p(T) = \Sigma_i \alpha_i T^i \neq 0$ and $p(T) v = 0$. Thus $v \in T(V)$.

Def. 1.4 If nonzero $R$ module $M$ only have submodules $\{0\}$ and $M$, $M$ is called a simple module, or irreducible module.

Let $M$ be an $R$ module, $S \subseteq R$ be a subset of $R$, $X \subseteq M$ a subset of $M$, then we define $S$-linear combinations of $X$ as

$$SX = \{ \Sigma_{i=1}^n s_i x_i,\ s_i \in S,\ x_i \in M \}$$

Prop 1.2 Let $\phi \neq X \subseteq M$ be a subset of $R$ module $M$, then $RX$ is submodule of $M$, called submodule generated by $X$ and denoted as $(X)$.

Prop 1.3 For $X \subseteq M$, $RX = (X) = \bigcap_{X \subseteq N,\, N \text{ submodule}} N$

Proof. Step 1 $RX \subseteq \bigcap_{X \subseteq N,\, N \text{ submodule}} N$, since $X \subseteq N$, $RX \subseteq N$.

Step 2 $\bigcap_{X \subseteq N,\, N \text{ submodule}} N \subseteq RX$, since $RX$ is a submodule and $X \subseteq RX$.

Def 1.5 If $M = RX$, $X$ is called the set of generators of $M$. If $X$ is finite, and $RX = M$, $M$ is called finitely generated. If $M = (x)$, $M$ is called a cyclic module.

Def 1.6 Let $\{N_i \mid i \in I\}$ be a family of submodules of $M$, we define

$$\Sigma_{i \in I} N_i := (\cup_{i \in I} N_i) = \{ y_{i_1} + \cdots + y_{i_k} \mid y_{i_j} \in N_{i_j},\ i_j \in I \}.$$

Def 1.7 Let $K$ be a submodule of $M$, consider the coset

$$M/K = \{ x + K \mid x \in M \}.$$

The addition is defined as $(x+K) + (y+K) := x+y+K$, the scalar product is defined as $a(x+K) = ax + K$. Then $M/K$ is a module called quotient module.