Chapter 1 Module

1.2 Exercise

2.1 Let M be an R module, prove that the following statements are equivalent:

(1) $M = 0$.

(2) For any R module N, there is unique module map from N to M.

(3) For any R module N, there is unique module map from M to N.

Proof. This is to prove zero module is a zero object, viz., it's initial and terminal.

"(1) ⇒ (2)": obvious.

"(2) ⇒ (1)": Suppose $M \neq 0$. Take $N = M$. $id: M \to M$ and $0: M \to M$ are different module maps, thus we arrive at a contradiction.

"(1) ⇒ (3)": Obvious.

"(3) ⇒ (1)": similar as "(2) ⇒ (1)".

2.2 Prove the equivalence of following statements:

(1) M is simple module.

(2) Any nonzero module map $M \to N$ is monomorphism.

(3) Any nonzero module map $N \to M$ is epimorphism.

Proof. "(1) ⇒ (2)": If $f: M \to N$ is not monomorphic, $\operatorname{Ker} f$ will be a nonzero submodule of M, thus a contradiction.

"(2) ⇒ (1)": Suppose M is not simple and N is a submodule $(\neq 0, M)$. Then quotient map $q: M \to M/N$ has $\ker q = N$, thus not monomorphic, we arrive at a contradiction.

"(1) ⇒ (3)": Similar to "(1) ⇒ (2)".

"(3) ⇒ (1)": Suppose M is not simple, $N \subsetneqq M$, $N \neq 0$, $N \hookrightarrow M$ is a module map, but it's not epimorphic.

2.3 (Schur's lemma) If M, N are simple modules, then any nonzero module map from M to N is an isomorphism. If M is simple, then $\operatorname{End}_R(M) = \operatorname{Hom}_R(M, M)$ is a division ring.

Proof. This is a result of Exercise 2.2.

2.4 For module map $f: M \to N$:

(1) If f is monomorphism, then $\operatorname{Ann}(M) \supseteq \operatorname{Ann}(N)$.

(2) If f is epimorphism, then $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(N)$.

Proof. (1) For monomorphism $M \xrightarrow{f} N$. If $r \in Ann(N)$, $r \cdot N = 0$. For any $m \in M$, $f(rm) = rf(m) = 0$.

Thus $rm \in Ker f = 0$, thus $r \in Ann(M)$.

(2) For epimorphism $M \xrightarrow{f} N$, if $r \in Ann(M)$, for any $n \in N$, there is $m \in M$ s.t. $f(m) = n$.

$r \cdot n = r \cdot f(m) = f(r \cdot m) = 0$. Since n is arbitrary, we see $r \in Ann(N)$.

2.5 Let $f: M \to N$ be an epimorphism and K be a submodule of M. Show:

(1) If $K \cap Ker f = 0$, then $f|_K : K \to N$ is monomorphic.

(2) If $K + Ker f = M$, then $f|_K : K \to N$ is epimorphic.

Proof. (1) $Ker(f|_K) = K \cap Ker f$.

(2) Use isomorphism theorem $(A+B)/B \cong A/(A \cap B)$.

Here $f: M \to N$ is epimorphic, $N \cong M/Ker f$, define quotient map

$q: M/Ker f \to N$. Since $M = K + Ker f$, $M/Ker f = (K + Ker f)/Ker f \cong K/(K \cap Ker f) \cong N$.

2.6 Prove that for R module M, we have R module isomorphism $Hom_R(R, M) \cong M$.

Proof. Define a map $\Psi: Hom_R(R, M) \to M$, $f \mapsto f(1)$. It satisfies $\Psi(f+g) = \Psi(f) + \Psi(g)$ and $\Psi(rf)$

$= r\Psi(f)$. Thus Ψ is a module map.

Set $f(1) = m$, for any $m \in M$, $f(r) = f(r \cdot 1) = r \cdot f(1) = r \cdot m$. f is a module map, thus $f \in Hom_R(R, M)$.

This means that Ψ is surjective.

$Ker \Psi = 0$ is clear.

2.7 Let A be a $\mathbb{Z}$ module. For $\mathbb{Z}$ module $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z} = \{\bar{0}, \bar{1}, \cdots, \overline{m-1}\}$ prove:

(1) $Hom_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m] := \{a \in A \mid m \cdot a = 0\}$.

(2) Use (1) to show that $Hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{(m,n)}$, where $(m,n) = gcd(m,n)$.

Proof. (1) Define module map $\Psi: f \mapsto f(\bar{1})$ (check it by yourself). Since $m \cdot f(\bar{1}) = f(m \cdot \bar{1}) = 0$.

$Im \Psi \subseteq A[m]$. For any $x \in A[m]$, we define $g_x(\bar{1}) = x$, it's clear that g_x is in $Hom_{\mathbb{Z}}(\mathbb{Z}_m, A)$.

This implies $A[m] \subseteq Im \Psi$.

To show $Ker \Psi = 0$, consider $f \in Ker \Psi$, $f(\bar{1}) = 0$, this implies $f(\bar{k}) = f(k \cdot \bar{1}) = k \cdot f(\bar{1}) = 0$. Thus $f = 0$.

(2) We need to show $\{a \in \mathbb{Z}_n \mid m \cdot a = 0\} \cong \mathbb{Z}_{(m,n)}$. Let $d = gcd(m,n)$ and $n = n_1 \cdot d$. We see

$$\{\bar{0}, \overline{n_1}, \overline{2n_1}, \cdots, \overline{(d-1) \cdot n_1}\}$$

is anihilated via the action of m. (Since $m = m_1 \cdot d$, $m \cdot k n_1 = m_1 k \cdot n$.)

The above module is isomorphic to $\mathbb{Z}_{(m,n)}$.

(Notice: $m = m_1 d$, $n = n_1 d$, and m_1, n_1 coprime. For $m \cdot \bar{a} = \bar{0}$, we have $n \mid ma$.

Thus $n_1 d \mid m_1 d a$, this implies $n_1 \mid m_1 a$, since $n_1 \nmid m_1$, we have $n_1 \mid a$.)

2.8 Determine $Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_n)$, $Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z})$, $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$, $Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$ and $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$.

(1) $Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_n) \cong \mathbb{Z}_n$ (by Exercise 2.6)

(2) $Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) \cong \{0\}$. (by Exercise 2.7)

(3) $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong \{0\}$.

For $f \in Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$, f is an Abelian group homomorphism. Consider $Im f \subseteq \mathbb{Z}$, suppose that n is the smallest positiven integer in $Im f$. There is $\frac{s}{t} \in \mathbb{Q}$ s.t. $f(\frac{s}{t}) = n$. This implies that $f(\frac{s}{2t}) + f(\frac{s}{2t}) = f(\frac{s}{t}) = n$. Let $f(\frac{s}{2t}) = m$, we have $2m = n$. m must be smaller than n. This is a contradiction.

(4) $Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Q}$ (by exercise 2.6)

(5) $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$.

By (4) $\mathbb{Q} \cong Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$, we only need to show $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \cong Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$.

Define $\Psi: Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$ by restriction of $f: \mathbb{Q} \to \mathbb{Q}$ to $f|_{\mathbb{Z}}: \mathbb{Z} \to \mathbb{Q}$.

"$Ker \Psi = 0$": If $\Psi(f) = f|_{\mathbb{Z}} = 0$. $f(1) = 0$. This implies that $t \cdot f(\frac{s}{t}) = f(s) = s \cdot f(1) = 0$.

Thus $f = 0$.

"$Im \Psi = Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$". For any $g \in Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q})$, define $\tilde{g}(\frac{s}{t}) = q$ s.t. $t \cdot q = g(s)$. We need to show $\tilde{g} \in Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$. $\tilde{g}(\frac{s}{t} + \frac{a}{b}) = \tilde{g}(\frac{sb+at}{tb}) = \frac{1}{tb} \cdot g(sb+at) = \frac{1}{tb}(b\, g(s) + t\, g(a)) = \frac{1}{t} g(s) + \frac{1}{b} g(a)$

$= \tilde{g}(\frac{s}{t}) + \tilde{g}(\frac{a}{b})$.

2.9. Let M, N be $\mathbb{Z}$ module. $Ann(M) = m\mathbb{Z}$, $Ann(N) = n\mathbb{Z}$, $Ann(Hom_{\mathbb{Z}}(M, N)) = d\mathbb{Z}$. Prove that d divides $\gcd(m, n)$. (Recall that $\mathbb{Z}$ is PID, every ideal is generated by a single element).

Proof. We need to show $\gcd(m,n) \in Ann(Hom_{\mathbb{Z}}(M, N))$. Since there exist $a, b \in \mathbb{Z}$ such that $\gcd(m,n) = am + bn$. For any $f \in Hom_{\mathbb{Z}}(M, N)$, we have $(am+bn) f(x) = am f(x) + bn f(x)$ $= f(am x) + bn f(x)$. $am \in Ann(M)$ implies $am x = 0$. $bn \in Ann(N)$ implies that $bn f(x) = 0$. Thus $(am+bn) f(x) = 0$ for all x. Since f is arbitrary, we see $\gcd(m,n) \in Ann(Hom_{\mathbb{Z}}(M,N))$.

2.10 Let R be an integral domain.

(1) For R module map $f: M \to N$, prove that $f(T(M)) \subseteq T(N)$ with $T(M)$ and $T(N)$ being torsion submodules. This means the restriction $f_T: T(M) \to T(N)$ is module map.

(2) If $0 \to K \xrightarrow{f} M \xrightarrow{g} N$ exact, then $0 \to T(K) \xrightarrow{f_T} T(M) \xrightarrow{g_T} T(N)$ exact.

(3) Give a counterexample that $T(M) \xrightarrow{g_T} T(N) \to 0$ is not exact even when $M \xrightarrow{g} N \to 0$ exact.

Proof. (1) If $x \in T(M)$, $\mathrm{Ann}(x) \neq 0$. $\exists r \in R$ s.t. $rx = 0$. Thus $r \cdot f(x) = f(rx) = 0$, implies $f(x) \in T(N)$.

(2) Step 1. $\mathrm{Ker} f_T = \mathrm{Ker} f \cap T(K) = 0$ since $\mathrm{Ker} f = 0$.

Step 2. Show $\mathrm{Im} f_T = \mathrm{Ker}\, g_T = \mathrm{Ker}\, g \cap T(M) = \mathrm{Im} f \cap T(M)$

For any $m \in \mathrm{Im} f \cap T(M)$. $\exists\, 0 \neq r \in R$ s.t. $rm = 0$. and $\exists\, k \in K$ s.t. $f(k) = m$. Then $r \cdot f(k) = f(r \cdot k) = 0$. Since $\mathrm{Ker} f = 0$. $r \cdot k = 0 \Rightarrow k \in T(K)$.

This means $\mathrm{Im} f \cap T(M) \subseteq \mathrm{Im} f_T$. The other direction is obvious.

(3) For $\mathbb{Z} \xrightarrow{q} \mathbb{Z}/6\mathbb{Z} \longrightarrow 0$. $T(\mathbb{Z}) = 0$, $T(\mathbb{Z}_6) = \mathbb{Z}_6$

$T(\mathbb{Z}) \to T(\mathbb{Z}/6\mathbb{Z}) \to 0$ is not exact.

2.12. (1) If $0 \to A \to B \xrightarrow{f} C \to 0$

$0 \to C \xrightarrow{g} D \to E \to 0$ are exact, then

$0 \to A \to B \xrightarrow{gf} D \to E \to 0$ is exact.

(2) Every exact sequence can be composed from short exact sequences from (1).

Proof.

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{\psi} E \to 0$$

① $\mathrm{Im} f = C$ thus $\mathrm{Im}\, gf = \mathrm{Ker}\, \psi$

② $\mathrm{Ker}\, g = 0$ thus $\mathrm{Ker}\, gf = f^{-1} \circ g^{-1}(0) = f^{-1}(0) = \mathrm{Ker} f = \mathrm{Im}\, \varphi$.