Chapter 1 Module

§1.2 Module homomorphism

- Definition and property of module homomorphism.
- Hom set $\mathrm{Hom}(M, M')$
- Isomorphism theorem of modules.
- Exact sequence.
- Short five Lemma, five lemma, snake lemma.

I. Module homomorphism

Def 2.1 A map $f: M \to M'$ between two R modules are called modue homomorphism if

1. $f(x+y) = f(x) + f(y)$, $\forall x, y \in M$.
2. $f(a\cdot x) = a \cdot f(x)$, $\forall a \in R$, $\forall x \in M$.

Or equivalently, $f(ax+by) = af(x) + bf(y)$, $\forall a, b \in R$, $\forall x, y \in M$.

Def 2.2 Module homomorphism $f: M \to M'$ is called:

1. Monomorphism if f is injective.
2. Epimorphism if f is surjective.
3. Isomorphism if f is bijective. In this case, M and M' are called isomorphic $M \cong M'$.

E.g. For submodule $K \subseteq M$, $\nu: M \to M/K$, $x \mapsto \bar{x} = x+K$ is epimorphism, it's called canonical homomorphism.

Def 2.3 For R module homomorphis $f: M \to M'$:

1. $\mathrm{Ker} f := \{x \in M \mid f(x) = 0\} = f^{-1}(0)$.
2. $\mathrm{Im} f := \{f(x) \in M' \mid x \in M\}$.
3. $\mathrm{Coker} f := M'/\mathrm{Im} f$.
4. $\mathrm{Coim} f := M/\mathrm{Ker} f$.

You need to check than $\mathrm{Ker} f$ and $\mathrm{Im} f$ are submodules.

Prop 2.1 For module homomorphism $f: M \to M'$, we have the following equivalent statements:

1. f is monomorphism
2. $\mathrm{Ker} f = 0$

3. For any R module K and module homomorphisms $g, h: K \to M$, $fg = fh \Rightarrow g = h$. Namely f can be cancelled from the left.

4. For any R module K and module homomorphism $g: K \to M$, $fg = 0 \Rightarrow g = 0$.

<u>Prop</u> 2.2 Let $f: M \to M'$ be a R module homomorphism, then the following statements ar equivalent:

1. f is epimorphism.
2. $\mathrm{Im} f = M'$.
3. For any R module K and module homomorphism $g, h: M' \to K$, $gf = hf \Rightarrow g = h$.
4. For any R module K and module homomorphism $g: M' \to K$, $gf = 0 \Rightarrow g = 0$.

<u>Prop</u> 2.3 Let M, M' be R modules, $f: M \to M'$ be a module homomorphism, then f is isomorphic iff there are $g, h: M' \to M$ such that $fg = 1_{M'}$ and $hf = 1_M$. In this case, $g = h$ and it's R module homomorphism.

II. Hom(M, M') as a module.

<u>Def.</u> Let M, M' be modules, $\mathrm{Hom}(M, M')$ is defined as set of all module homomorphisms between M and M'. The addition $f + g$ is defined as $(f+g)(x) := f(x) + g(x)$ for all $x \in M$. The scalar product is defined as $(a \cdot f)(x) := a f(x)$ $\forall a \in R$, $\forall x \in M$. This makes $\mathrm{Hom}(M, M')$ a R module.

When $M = M'$, we also set $\mathrm{End}(M) := \mathrm{Hom}(M, M)$. $\mathrm{End}(M)$ is also a ring with multiplication defined by composition of maps. Notice that composition bilinear: $(af + bg) \cdot h = a fh + b gh$, $h(af + bg) = ahf + bhg$.

III. Isomorphism theorem

<u>Thm</u> 2.4 Let $f: M \to M'$ and $g: M \to N$ be module homomorphisms, and $g: M \twoheadrightarrow N$ is epimorphism such that $\mathrm{Ker}\, g \subseteq \mathrm{Ker} f$. Then there is unique $h: M' \to M$ such that $f = hg$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g"'] & M' \\ & N \arrow[u, dashed, "\exists ! h"'] \end{tikzcd}$$

Moreover, $\mathrm{Ker}\, h = g(\mathrm{Ker} f)$, $\mathrm{Im}\, h = \mathrm{Im} f$. Thus h is monomorphism iff $\mathrm{Ker}\, g = \mathrm{Ker} f$; h is epimorphism iff f is epimorphism.

Proof. For $n \in N$, since $g: M \to N$ is epimorphism, there is $m \in M$ s.t. $g(m) = n$.

We set $h(n) = f(m)$, it's clear that $f = hg$. But we need to show h is well-defined. Suppose that $m \neq m'$ s.t. $g(m) = g(m') = n$. $g(m-m') = 0 \Rightarrow m-m' \in \operatorname{Ker} g \subseteq \operatorname{Ker} f$

This implies $f(m) = f(m')$. Thus $h(m) = h(m')$, h is well-defined.

To show h is module homomorphism, for $n_1, n_2 \in N$, $\exists\, m_1, m_2 \in M$ s.t. $g(m_1) = n_1$, $g(m_2) = n_2$.

$h(am_1 + bm_2) = f(am_1 + bm_2) = af(m_1) + bf(m_2) = ah(n_1) + bh(n_2)$.

Thm 2.5 1. Let f be a R module homomorphism, then $M/\operatorname{Ker} f \cong \operatorname{Im} f$.

2. Let $K \subseteq N$ both be submodules of M, then $M/N \cong (M/K)/(N/K)$.

3. Let K, N be submodules of M, then $(N+K)/K \cong N/(N \cap K)$.

Proof. 1. Set $N = M/\operatorname{Ker} f$, $g: M \to N$ as canonical quotient map. and $f: M \twoheadrightarrow \operatorname{Im} f$.

Thm 2.4 directly implies that there is a isomorphism $h: N \to \operatorname{Im} f$.

2. Define $f: M/K \to M/N$, $x + K \mapsto x + N$. Since $K \subseteq N$, $x + K = x' + K$ implies $x - x' \in K \subseteq N$. Thus $x + N = x' + N$, f is well-defined. It's also clear that f is module homomorphism. $\operatorname{Ker} f = N/K$, $\operatorname{Im} f = M/N$. Thus 1 implies $\operatorname{Im} f = (M/K)/(N/K)$.

3. Define $f: N \to (N+K)/K$, $x \mapsto x + K$, use 1.

<u>Coro</u> 2.6 Let $f: M \to M'$ be epimorphism, then

$$N \mapsto f(N) = \{f(x) \mid x \in N\}$$

$$N' \mapsto f^{-1}(N') = \{x \in M \mid f(x) \in N'\}$$

establish a one-to-one correspondence beween submodules of M that contain $\operatorname{Ker} f$ and submodules of M'.

<u>Prop</u> 2.7 R module M is a cyclic module iff M is isomorphic to a quotient of R module R. If x is a generator of M, then $M \cong R/\operatorname{Ann}_R(x)$. M is simple iff $\operatorname{Ann}_R(x)$ is a maximal ideal.

Proof. First statement.

"$\Rightarrow$" Suppose M is cyclic, viz., $M = (x)$. Then we define R-module map

$$\rho: r \mapsto rx$$

It's clear that ρ is epimorphism. $\operatorname{Ker} \rho = \{r \in R \mid rx = 0\} = \operatorname{Ann}_R(x)$. From thm 2.5

$M = \operatorname{Im} \rho = R/\operatorname{Ker} \rho = R/\operatorname{Ann}_R(x)$.

"$\Leftarrow$" Let $I \trianglelefteq R$ be an ideal, then $R/I = \{\bar{r} = r + I \mid r \in R\}$ is a quotient module.

Notice that $R/I = (\bar{1})$, i.e., R/I is cyclic.

Second statement.

"$\Rightarrow$" M is simple, suppose $Ann_R(x)$ is not maximal ideal, there will be an ideal $Ann_R \subsetneq I \neq M$.

This means R/I is a submodule of $R/Ann_R(x)$. This is a contradiction. (Corollary 2.6)

"$\Leftarrow$" Similarly, based on Corollary 2.6.

IV. Exact sequences.

Def 2.4. Consider R module homomophism

$$M' \xrightarrow{f} M \xrightarrow{g} M''.$$

If $\operatorname{Im} f = \operatorname{Ker} g$, we call f and g are exact at M. For a finite or infinite

$$\cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \longrightarrow \cdots$$

if for every M_n, $\operatorname{Im} f_n = \operatorname{Ker} f_{n+1}$, we call it an exact sequence.

Prop 2.8 1. $0 \to M \xrightarrow{f} N$ is exact iff f is monomorphism.

2. $M \xrightarrow{f} N \to 0$ is exact iff f is epimorphism

3. $0 \to M \xrightarrow{f} N \to 0$ is exact iff f is isomorphism.

By definition of $\ker f$ and $\operatorname{Coker} f$, the following sequence is exact:

$$0 \to \ker f \xhookrightarrow{i} M \xrightarrow{f} N \xrightarrow{\nu} \operatorname{Coker} f \to 0.$$

Similary, f is monomorphic iff the following sequence is exact:

$$0 \to M \xrightarrow{f} N \xrightarrow{\nu} \operatorname{Coker} f \to 0.$$

f is epimorphic iff the following sequence is exact:

$$0 \to \ker f \xhookrightarrow{i} M \xrightarrow{f} N \to 0.$$

The following exact sequence is called a short exact sequence:

$$0 \to K \xhookrightarrow{f} M \xrightarrow{g} N \to 0.$$

We can treat K as a submodule of M and N a quotient module of M. This short exact sequence is also called an extension of N via K.

Lemma 2.9 (Short five lemma) Consider the following commutative diagram of R module homomorphisms

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & N \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & K' \arrow[r, "f'"] & M' \arrow[r, "g'"] & N' \arrow[r] & 0
\end{tikzcd}$$

Two horizontal sequences are assumed to be exact, then

1. If α, γ are monomorphisms then β is also monomorphism.
2. If α, γ are epimorphism, then β is also epimorphism.
3. If α, γ are isomorphisms, then β is also isomorphism.

Proof. Diagram chasing!!

1. We need to show that $\operatorname{Ker}\beta = \{0\}$. Suppose $m \in M$ s.t. $\beta(m) = 0$, we need to show $m = 0$.

   $\gamma g(m) = g'\beta(m) = g'(0) = 0$.

   Since γ is monomorphic, $g(m) = 0$. Thus $m \in \operatorname{Ker} g = \operatorname{Im} f$, $\exists\, k \in K$ s.t. $f(k) = m$.

   $f'\alpha(k) = \beta f(k) = \beta(m) = 0$.

   Since f' is monomorphic, $\alpha(k) = 0$. α is also monomorphic, thus $m = 0$.

2. For $m' \in M'$, we need to show that there is $x \in M$ s.t. $\beta(x) = m'$.

   Let $m' \in M'$, $g(m') \in N$. Since γ is epimorphic, there is $n \in N$ s.t. $g'(m') = \gamma(n)$. Since g is epimorphic, there is $m \in M$ s.t. $g(m) = n$. Then we have

   $$g'(m') = \gamma(n) = \gamma g(m) = g'\beta(m).$$

   Thus $\beta(m) - m \in \operatorname{Ker} g' = \operatorname{Im} f'$, $\exists\, k' \in K'$ s.t. $f(k') = \beta(m) - m'$. Since α is epimorphic, $\exists\, k \in K$ s.t. $\alpha(k) = k'$. Take $m - f(k) \in M$, we have

   $$\beta(m - f(k)) = \beta(m) - \beta f(k).$$

   Notice that $\beta f(k) = f'\alpha(k) = f'(k') = \beta(m) - m'$, thus

   $$\beta(m - f(k)) = \beta(m) - \beta f(k) = m'.$$

   This means β is epimorphic.

3. This is a result of 1 and 2.

Lemma 2.10 (five lemma) Consider the following commutative diagram with exact row

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "\alpha"] & B \arrow[r, "f_2"] \arrow[d, "\beta"] & C \arrow[r, "f_3"] \arrow[d, "\gamma"] & D \arrow[r, "f_4"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "g_1"'] & B' \arrow[r, "g_2"'] & C' \arrow[r, "g_3"'] & D' \arrow[r, "g_4"'] & E'
\end{tikzcd}$$

1. If α is epimorphism and β, δ are monomorphisms, then γ is monomorphic.
2. If ε is monomorphism, and β, δ are epimorphisms, then γ is epimorphic
3. If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then γ is isomorphism. (If β, γ are isomorphisms, α is epimorphism and ε is monomorphism, then γ is isomorphism.)

Proof. This can be derived from snake lemma. It can also be proved by diagram chasing.

Lemma 2.11 (Snake lemma) Consider the following commutative diagram (black one)

$$\begin{tikzcd}
 & \operatorname{Ker}\alpha \arrow[r] \arrow[d] & \operatorname{Ker}\beta \arrow[r] \arrow[d] & \operatorname{Ker}\gamma \arrow[d] \arrow[dddll, "\delta"] & \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{Coker}\alpha \arrow[r] & \operatorname{Coker}\beta \arrow[r] & \operatorname{Coker}\gamma &
\end{tikzcd}$$

where two rows are exact, then there exist R module connecting homomorphism

$$\delta : \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$$

such that the following sequence is exact:

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma .$$

If f is monomorphism, then so is $\operatorname{Ker}\alpha \to \operatorname{Ker}\beta$; and if g is epimorphism, then so is $\operatorname{Coker}\beta \to \operatorname{Coker}\gamma$.