Chapter 1 Module

§1.3 Direct Sum and Direct Product of Modules

- Direct sum and splitting lemma.
- Product and coproduct in ${}_R\mathrm{Mod}$.

(I) Direct sum and splitting lemma

Def 3.1. Let M_1, M_2 be two R modules, for $M = M_1 \times M_2 = \{(x_1, x_2) \mid x_1 \in M_1, x_2 \in M_2\}$, define

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$

$$a(x_1, x_2) = (ax_1, ax_2).$$

Then M becomes a R module, which is called direct sum of M_1 and M_2 and denoted as $M_1 \oplus M_2$.

For direct sum $M_1 \oplus M_2$ we can define:

1. $\iota_1 : M_1 \to M_1 \oplus M_2$, $x_1 \mapsto (x_1, 0)$; $\iota_2 : M_2 \to M_1 \oplus M_2$, $x_2 \mapsto (0, x_2)$

   They are monomorphisms, thus embeddings.

2. $\pi_1 : M_1 \oplus M_2 \to M_1$, $(x_1, x_2) \to x_1$; $\pi_2 : M_1 \oplus M_2 \to M_2$, $(x_1, x_2) \mapsto x_2$

   They are epimorphisms.

Thm. It is easy to check that, we have the following equlities:

① $\pi_1 \iota_1 = id_{M_1}$, $\pi_2 \iota_2 = id_{M_2}$.

② $\iota_1 \pi_1 + \iota_2 \pi_2 = id_{M_1 \oplus M_2}$.

③ $\pi_1 \iota_2 = 0$, $\pi_2 \iota_1 = 0$.

Notice that ① and ② means that $M_1 \oplus M_2$ is a biproduct (a product and a coproduct in the category ${}_R\mathrm{Mod}$. ③ can be derived from ① and ②.

Proof. • $(M_1 \oplus M_2, \pi_1, \pi_2)$ is a product.

• $(M_1 \oplus M_2, \iota_1, \iota_2)$ is a coproduct.

• $\iota_1 = id_{M_1 \oplus M_2} \iota_1 = (\iota_1 \pi_1 + \iota_2 \pi_2)\iota_1 = \iota_1 \pi_1 \iota_1 + \iota_2 \pi_2 \iota_1 = \iota_1 id_{M_1} + \iota_2 \pi_2 \iota_1$
$= \iota_1 + \iota_2 \pi_2 \iota_1$

Since Hom set is an Abelian group, we have $\iota_2 \pi_2 \iota_1 = 0$. This implies $\pi_2 \iota_2 \pi_2 \iota_1 = 0$.
Thus $id_{M_2} \pi_2 \iota_1 = 0 = \pi_2 \iota_1$. Similarly, we have $\pi_1 \iota_2 = 0$.

<u>Internal direct sum</u>: For a module M, M_1, M_2 are two submodules such that:

(1) $M_1 \cap M_2 = 0$

(2) $M_1 + M_2 = M$

Define $\Psi: M_1 \oplus M_2 \to M$, $(x_1, x_2) \mapsto x_1 + x_2$. Since (2), Ψ is surjective. To show $\ker \Psi = 0$, consider $(x_1, x_2) \in \ker \Psi$, we have $x_1 + x_2 = 0$. This implies $x_1 = -x_2 \in M_1 \cap M_2 = 0$. Thus $x_1 = x_2 = 0$. Therefore $M_1 \oplus M_2 \cong M$. M is called internal direct sum of M_1, M_2. For any $x \in M$, there is a unique decomposition $x = x_1 + x_2$.

<u>Lemma</u> 3.1 Let $f: N \to M$, $g: M \to N$ be module maps such that $gf = id_N$. Then f is monomorphism, g is epimorphism and $M = \operatorname{Im} f \oplus \operatorname{Ker} g$.

$$\begin{tikzcd} N \arrow[r, tail, "f"] \arrow[rr, bend right, "id_N"'] & M \arrow[r, two heads, "g"] & N \end{tikzcd}$$

In this case f, g are called split.

Remark. Recall f is monomorphic if it can be cancelled from left: $fg = fh \Rightarrow g = h$. Here f is split means f has left inverse, it is a stronger constraint. Similarly for g, g is split epimorphic means g has right inverse, which is a stronger constraint.

Proof. • f has left inverse, thus f is injective. (left inverse is called section)

• g has right inverse, thus g is surjective. (right inverse is called retract)

To show that $M = \operatorname{Im} f \oplus \operatorname{Ker} g$. We need to show

(1) $M = \operatorname{Im} f + \operatorname{Ker} g$

Suppose $x \in M$, $x = fg(x) + (x - fg(x))$. $fg(x) \in \operatorname{Im} f$. To show $x - fg(x) \in \operatorname{Ker} g$, notice $g(x - fg(x)) = g(x) - gfg(x) = g(x) - id \circ g(x) = 0$.

(2) $\operatorname{Im} f \cap \operatorname{Ker} g = 0$.

Suppose $x \in \operatorname{Im} f \cap \ker g$, $\exists\, y \in N$ s.t. $x = f(y)$, $g(x) = 0 = gf(y) = y$. Thus $f(y) = x = 0$.

<u>Def</u> 3.2. If f has left inverse, we call f a slitpmonomorphism. If g has a right inverse, we call g a split epimorphism. If in the short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

both f and g are split, then the sequence is called a split short exact sequence.

<u>E.g.</u> The sequence $0 \to M_1 \xrightarrow{\iota_1} M_1 \oplus M_2 \xrightarrow{\pi_2} M_2 \to 0$ is split short exact sequence.

<u>Prop</u> 3.2 (Splitting lemma). Consider the R module short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$$

The following statements are equivalent:

(1) The sequence is split.

(2) $f: M_1 \to M$ is split monomorphic.

(3) $g: M \to M_2$ is split epimorphic.

(4) $\mathrm{Im} f = \mathrm{Ker}\, g$ is direct summand of M.

(5) Every $h: M_1 \to N$ decompose through f:

$$\begin{tikzcd} & & N \\ 0 \arrow[r] & M_1 \arrow[ur, "h"] \arrow[r, "f"'] & M \arrow[u, dotted, "\exists \bar{h}"'] \end{tikzcd}$$

(6) Every $h: N \to M_2$ decompose through g:

$$\begin{tikzcd} M \arrow[r, "g"] & M_2 \arrow[r] & 0 \\ N \arrow[u, dotted, "\exists \bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

(7) There is isomorphism between short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "id_{M_1}"] & M \arrow[r, "g"] \arrow[d, "\phi"] & M_2 \arrow[r] \arrow[d, "id_{M_2}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r, "\iota_1"] & M_1 \oplus M_2 \arrow[r, "\pi_2"] & M_2 \arrow[r] & 0 \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2) (1) $\Rightarrow$ (3) are results of definition

(7) $\Rightarrow$ (1) :

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "id_{M_1}"'] & M \arrow[d, "\cong\ \phi"] & \\ M_1 \arrow[r, "\iota_1"] \arrow[rr, bend right, "= id_{M_1}"'] & M_1 \oplus M_2 \arrow[r, "\pi_1"] & M_1 \end{tikzcd} \Rightarrow f \text{ split}$$

$$\begin{tikzcd} & M \arrow[r, "g"] & M_2 \arrow[d, "id_{M_2}"] \\ M_2 \arrow[r, "\iota_2"] \arrow[rr, bend right, "= id_{M_2}"'] & M_1 \oplus M_2 \arrow[u, "\cong\ \phi^{-1}"] \arrow[r, "\pi_2"] & M_2 \end{tikzcd} \Rightarrow g \text{ split}$$

(2) $\Rightarrow$ (4) , (3) $\Rightarrow$ (4) are direct result of lemma 3.1.

(4) $\Rightarrow$ (5) ; Suppose $M = \mathrm{Im} f \oplus K$. Then for any $m \in M$, $\exists\, m_1 \in M_1$ and $k \in K$ s.t. $m = f(m_1) + k$.
Define $\bar{h}: m = f(m_1) + k \mapsto h(m_1)$. $\bar{h}$ is well-defined since $\mathrm{Ker} f = 0$.

$m+n = f(m_1)+k_1+f(n_1)+k_2 = f(m_1+n_1)+(k_1+k_2) \Rightarrow \bar{h}(m+n) = \bar{h}(m)+\bar{h}(n)$.

$rm = rf(m_1)+rk = f(rm_1)+rk \Rightarrow \bar{h}(rm) = r\bar{h}(m)$.

Thus $\bar{h}$ is R module map. $\bar{h}f(x) = \bar{h}(f(x)+0) = h(x)$.

$(4)\Rightarrow(6)$: Suppose $M = \operatorname{Ker} g \oplus K$. Since $\operatorname{Ker} g \cap K = 0$ and $g(M) = g(K)$, we see that $g|_K : K \to M_2$ is isomorphism. Then we define $\bar{h} = g^{-1}h$.

$(5)\Rightarrow(2)$ Consider the following diagram

$$\begin{tikzcd} & & M_1 \\ 0 \arrow[r] & M_1 \arrow[ur, "id_{M_1}"] \arrow[r, "f"'] & M \arrow[u, dotted, "\bar{h}"'] \end{tikzcd}$$

$(6)\Rightarrow(3)$ Consider the following diagram

$$\begin{tikzcd} M \arrow[r, "g"] & M_2 \arrow[r] & 0 \\ M_2 \arrow[u, dotted, "\exists \bar{h}"] \arrow[ur, "id_{M_2}"'] & & \end{tikzcd}$$

$(2)\Rightarrow(7)$ Here we need to use short five lemma

Suppose $h: M \to M_1$ satisfies $hf = id_{M_1}$, then define $\phi: M \to M_1 \oplus M_2$, $m \mapsto (h(m), g(m))$.

It's clear that ϕ is module map. $\phi f(m_1) = (hf(m_1), g(m_1)) = (m_1, 0) = \iota_1(m_1)$.

$\pi_2 \circ \phi(m) = g(m)$. Thus we have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r, "f"] \arrow[d, "id_{M_1}"] & M \arrow[r, "g"] \arrow[d, "\phi"'] & M_2 \arrow[r] \arrow[d, "id_{M_2}"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r, "\iota_1"'] & M_1 \oplus M_2 \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

From short five lemma, we see ϕ is isomorphism.

To sumarize, we now have

$$\begin{tikzcd} (7) \arrow[d] & & (5) \arrow[dl] \\ & (2) \arrow[ul] \arrow[dr] & \\ (1) \arrow[ur] \arrow[dr] \arrow[r, phantom, "①"] & \arrow[r, phantom, "②"] & (4) \arrow[uu] \arrow[dd] \\ & (3) \arrow[ur] \arrow[r, phantom, "③"] & \\ & & (6) \arrow[ul] \end{tikzcd}$$

① : $(1) \Leftrightarrow (2) \Leftrightarrow (7)$

② ③ : $(2) \Leftrightarrow (5) \Leftrightarrow (4) \Leftrightarrow (6) \Leftrightarrow (3)$

(II) Product and coproduct in ${}_R\mathrm{Mod}$

Def. (Product). For an index set I (finite or infinite), let $\{M_i\}_{i\in I}$ be a family of R modules. The product (direct product) is $M = \prod_{i\in I} M_i$ with elements $\{m_i\}_{i\in I}$, we define

① $\{m_i\}_{i\in I} + \{n_i\}_{i\in I} = \{m_i + n_i\}_{i\in I}$.

② $a\{m_i\}_{i\in I} = \{am_i\}$.

It's clear that $\prod_{i\in I} M_i$ is a module. For each $j\in I$, we define canonical projection

$$\pi_j : \prod_{i\in I} M_i \to M_j$$
$$\{m_i\}_{i\in I} \mapsto m_j.$$

All canonical projections are epimorphisms.

Prop (1) For any R module N and module maps $\{f_i : N \to M_i\}$, there exists a unique $\bar{f} : N \to \prod_{i\in I} M_i$ such that $f_j = \pi_j \bar{f}$ for all $j \in I$.

$$\begin{tikzcd} \prod_{i\in I} M_i \arrow[r, "\pi_i"] & M_i \\ N \arrow[u, dashed, "\exists!\bar{f}"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

(2) If there is a module M equipped with a set of module maps $\{p_i : M \to M_i\}_{i\in I}$ such that for any module N and module maps $\{f_i : N\to M_i\}$, there is a unique module map $\bar{f} : N \to M$ such that $f_j = p_j \bar{f}$ $\forall j\in I$. Then we have an isomorphism

$$\phi : M \rightarrowtail\!\!\!\!\twoheadrightarrow \prod_{i\in I} M_i$$

and $p_i = \pi_i \phi$

Def (Coproduct) For an index set I (finite or infinite), consider a subset of $\prod_{i\in I} M_i$

$$\coprod_{i\in I} = \{\{m_i\}_{i\in I} \in \prod_{i\in I} M_i \mid \text{all } m_i = 0 \text{ except for finite indices}\}.$$

This is called coproct of $\{M_i\}$, also called direct sum and denotes $\bigoplus_{i\in I} M_i$.

Define canonical injection

$$\iota_i : M_i \to \bigoplus_{i\in I} M_i$$
$$m_i \longmapsto \{\cdots, 0, m_j, 0, \cdots\}.$$

They are monomorphisms.

Prop (1) For any R module N and a family of R module homomorphisms $\{g_i : M_i \to N\}$, there exists unique $\bar{g} : \bigoplus_{i\in I} M_i \to N$ such that $g_j = \bar{g}\iota_j$ $\forall j \in I$

$$\begin{tikzcd} M_j \arrow[r, "\iota_j"] \arrow[dr, "g_j"'] & \bigoplus_{i\in I} M_i \arrow[d, dashed, "\exists!\,\bar{g}"] \\ & N \end{tikzcd}$$

(2) If R module M equipped with a set of module maps $\{i_j : M_j \to M\}$ such that for any R module N and $\{g_i : M_i \to N\}$, there exists unique $\tilde{g}$ s.t. $g_j = \tilde{g}\, i_j$ for all $j \in I$. Then there is isomorphism $\phi : M \rightarrowtail\!\!\twoheadrightarrow \bigoplus_{i\in I} M_i$ and $\iota_j = \phi\, i_j$ $\forall j \in I$.

(III) Properties in Hom set.

Prop. For R modules $\{M_i\}_{i\in I}$ and N, we have

$$\prod_{i\in I} \mathrm{Hom}(N, M_i) \cong \mathrm{Hom}(N, \prod_{i\in I} M_i).$$

Proof. For $\{f_i : N \to M_j\}$, there exist a unique $\bar{f} : N \to \prod_{i\in I} M_i$. Define $\phi(\{f_j\}) = \bar{f}$.

For $g \in \mathrm{Hom}(N, \prod_{i\in I} M_i)$, define $g_i = \pi_i \circ g$, we obtain a map from $\mathrm{Hom}(N, \prod_{i\in I} M_i)$ to $\prod_{i\in I} \mathrm{Hom}(N, M_i)$.

Prop. for R modules $\{M_i \mid i \in I\}$ and N, we have

$$\prod_{i\in I} \mathrm{Hom}(M_i, N) \cong \mathrm{Hom}(\bigoplus_{i\in I} M_i, N).$$

Proof. Define $\phi : \prod_{i\in I} \mathrm{Hom}(M_i, N) \to \mathrm{Hom}(\bigoplus_{i\in I} M_i, N)$

$$\{g_i\} \longmapsto \bar{g}$$

For any $h \in \mathrm{Hom}(\bigoplus_{i\in I} M_i, N)$, define $h_i = h\, \iota_i \in \mathrm{Hom}(M_i, N)$.