Chapter 1 Module

§1.4 Free Modules

- Definition and Examples.
- Properties and applications.

(I) Definition and examples

Def. 4.1 Let X be a subset of R module M, if for any finite $\{x_i\}_{i=1}^n \subseteq X$ and $a_i \in R$, we have

$$\sum_{i=1}^n a_i x_i = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0.$$

Then X is called linearly independent.

If X can generate M and X is linearly independent, X is called a basis of M.

If M has a basis, then M is called a free module.

Example. Every vector space (finite or inifinite dimensional) has a basis, thus they are all free modules.

(Try to prove it, you will need to use Zorn's lemma).

Example. Free $\mathbb{Z}$ modules are free Abelian group.

Prop 4.1 V is free R module iff V is inner direct sums of its cyclic submodules: $V = \oplus_{i\in I} V_i$, each V_i is cyclic and $V_i \cong R$. (Cyclic module $M \cong R/\mathrm{Ann}(x)$ is free iff $\mathrm{Ann}(x) = 0$, $M = \langle x \rangle \cong R$.)

Proof. "$\Rightarrow$" Since V is free module, suppose X is a basis of V, for each $x \in X$, define

$$V_x := \langle x \rangle = Rx.$$

Since X is linearly independent, $\mathrm{Ann}(x) = 0$. Thus $V_x \cong R/\mathrm{Ann}(x) = R$ (Prop 2.7).

① $\sum_{x\in X} V_x = V$ ② $V_x \cap (\sum_{y\neq x} V_y) = 0$

Thus $V = \oplus_{x\in X} V_x$.

"$\Leftarrow$" Suppose $V_i = \langle x_i \rangle = Rx_i$. Let $X = \{x_i\}$, we need to show X is a basis.

① $V = \langle X \rangle$ is clear.

② $\sum_i r_i x_i = 0 \Rightarrow r_i x_i = 0$ for all i (zero has unique decomposition in direct sum).

But $V_i \cong R/\mathrm{Ann}(x_i) = R$. We see $\mathrm{Ann}(x_i) = 0$. Thus $r_i = 0$.

Prop 4.2. Let V be a free module, there exists a set X and set map $\iota: X \longrightarrow V$ such that for any module M and set map $f: X \longrightarrow M$, there exists unique module map $\bar{f}: V \to M$ such that $f = \bar{f}\iota$.

Diagramatically:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & V \text{ (free)} \arrow[d, dotted, "\exists!\, \bar{f}"] \\ & M \end{tikzcd}$$

Proof. Choose X as basis of V, define $\bar{f}(\sum r_i x_i) = r_i f(x_i)$.

Prop. 4.3 Any R module is a quotient module of a free module. If M is finitely generated, then M is quotient module of finitely generated free module.

Proof. Let X be a generating set of M (always exists, since we can choose $X = M$), when M is finitely generated, then X can be chosen as a finite set.

Set free module $V = \oplus_{x \in X} R_x$ with cyclic module $R_x \cong R$ and g_x is the generating element.

Define $\iota(x) = g_x$, then we obtain a map $\iota: X \to V$.

Let $f: X \hookrightarrow M$ be embedding map. Then by prop 4.2, we have module map $\bar{f}: V \to M$.

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & V = \bigoplus_{x \in X} R_x \arrow[d, dotted, "\exists!\, \bar{f}"] \\ & M \end{tikzcd}$$

Since $X = f(X) \subseteq \bar{f}(V)$ and $M = RX$. We see $M \leq \bar{f}(V)$. $\bar{f}$ is surjective.

Prop 4.4. Let $f: M \to N$ be module epimorphism, V be a free module, $h: V \to N$ be a module map. Then there exists a not necessarily unique module map $\bar{h}: V \to M$ such that $h = f\bar{h}$.

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r] & 0 \\ V \arrow[u, dotted, "\exists!\, \bar{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Proof. Choose a basis X of V. $\iota: X \hookrightarrow V$ be embedding. Since $f: M \to N$ is surjective,

For any $x \in X$ we could find a $m_x \in M$ such that $f(m_x) = h(x)$. Define

$$g: X \to M, \quad x \mapsto m_x.$$

Then by prop 4.2, we have commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "g"'] & V \arrow[d, dotted, "\bar{g}"] \\ & M \end{tikzcd}$$

Notice $\bar{g}$ is a module map and $fg(x) = h(x)$, we have $f\bar{g}\iota = h\iota$. Since X is basis, $\bar{g}\iota$ can be extended into a module map $\bar{h}$.

Thm 4.5. Let V be a finitely generated free module, then all bases of V have the same number of elements.

Proof. This is the same as that for vector space.

Def. 4.2 Let R be a commutative ring, V is a finitely generated free module, then we call the number of elements in basis X as the rank of V, denotes $rk(V)$.

Remark. 1. Any free module V with $rk(V) = n$ is isomorphic to $\underbrace{R \oplus \cdots \oplus R}_{n}$.

2. $Hom_R(R^m, R^n) \cong M_{n,m}(R) = \{n \times m \text{ matrices}\}$ as additive group.

3. $Hom_R(R^n, R^n) \cong M_n(R)$ as ring.

Prop 4.6 When R is a PID, V is a free R module with $rk(V) = n$. Then any submodule N are free module and $rk(N) \leq n$. (Also holds for infinite rank modules).