Chapter 1 Module

§1.5 Hom and Projective Modules

- Hom functor
- Definition of projective modules
- Properties of projective module
- Dual modules

(I) Hom functor

Let ${}_R\mathrm{Mod}$ be the category of left R modules, recall that $\mathrm{Hom}_R(M,N)$ is a module. This means that we can introduce two functor

(1) For $M \in {}_R\mathrm{Mod}$, $\mathrm{Hom}_R(M,\bullet)$ is a functor from ${}_R\mathrm{Mod}$ to ${}_R\mathrm{Mod}$

Objects: $A \longmapsto \mathrm{Hom}_R(M,A)$

Maps: $(A \xrightarrow{f} B) \longmapsto (\mathrm{Hom}_R(M,A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(M,B))$

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ M \arrow[u, "h"] \arrow[ur, "\bar{f}(h) := fh"'] & \end{tikzcd} \qquad \bar{f} = \mathrm{Hom}_R(M,f) = f_*$$

(2) For $N \in M$, $\mathrm{Hom}(\bullet, N)$ is a contravariant functor from ${}_R\mathrm{Mod}$ to ${}_R\mathrm{Mod}$

Objects: $A \longmapsto \mathrm{Hom}_R(A,N)$

maps: $(A \xrightarrow{f} B) \longmapsto (\mathrm{Hom}(B,N) \xrightarrow{\tilde{f}} \mathrm{Hom}(A,N))$

$$\begin{tikzcd} & N \\ A \arrow[ur, "\tilde{f}(h)"] \arrow[r, "f"'] & B \arrow[u, "h"'] \end{tikzcd} \qquad \tilde{f} = \mathrm{Hom}_R(f,N) = f^*$$

Prop 5.1. The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact iff for any $M \in {}_R\mathrm{Mod}$, the sequence $0 \to \mathrm{Hom}_R(M,A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(M,B) \xrightarrow{\bar{g}} \mathrm{Hom}(M,C)$ is exact.

Proof. "$\Rightarrow$" ① $\bar{f}$ is monomorphic

Suppose $h \in \mathrm{Ker}\,\bar{f}$, then $\bar{f}(h) = fh = 0$. Since f is monomorphic, we see $h = 0$.

② $\mathrm{Im}\,\bar{f} \subseteq \mathrm{Ker}\,\bar{g}$. Since $gf = 0$, we have $\bar{g}\bar{f}(h) = gfh = 0$.

③ $\operatorname{Ker}\bar g \subseteq \operatorname{Im}\bar f$. Suppose $h \in \ker \bar g$, $\bar g(h) = gh = 0$. Thus $\operatorname{Im} h \subseteq \operatorname{Ker} g = \operatorname{Im} f$. Since f is monomorphic, it has inverse on $\operatorname{Im} f$. Thus $f^{-1}h$ is well-defined in $\operatorname{Hom}_R(M, A)$. Thus $h = \bar f(f^{-1}h) \in \operatorname{Im}\bar f$. (Notice that you need to show $f^{-1}: \operatorname{Im} f \to A$ is a module map. In fact f has two-side inverse on $\operatorname{Im} f$.)

"$\Leftarrow$" ① $\ker f = 0$: Take $M = \ker f$, and $i: \operatorname{Ker} f \hookrightarrow A$ be inclusion, then $\bar f(i) = fi = 0$ implies $i = 0$. Thus $\operatorname{Ker} f = 0$.

② $gf = 0$: Take $M = A$, $gf = gf\,\mathrm{id}_A = \bar g \bar f(\mathrm{id}_A) = 0(\mathrm{id}_A) = 0$.

③ $\operatorname{Ker} g \subseteq \operatorname{Im} f$: Take $M = \operatorname{Ker} g$, $j: \operatorname{Ker} g \hookrightarrow B$ be inclusion. $\bar g(j) = gj = 0$. Thus $j \in \ker \bar g = \operatorname{Im}\bar f$. There exists $h \in \operatorname{Hom}_R(\operatorname{Ker} g, A)$ such that $j = \bar f(h) = fh$. This implies that $\forall x \in \operatorname{Ker} g$, $\exists\, h(x) \in A$ s.t. $x = j(x) = f(h(x))$, thus $\ker g \subseteq \operatorname{Im} f$.

Prop 5.2. The sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact iff for any R module N, the following

$$0 \longrightarrow \operatorname{Hom}_R(C, N) \xrightarrow{\tilde g} \operatorname{Hom}_R(B, N) \xrightarrow{\tilde f} \operatorname{Hom}_R(A, N)$$

is exact.

Proof. "$\Rightarrow$" ① $\operatorname{Ker}\tilde g = 0$. Suppose $h \in \operatorname{Ker}\tilde g$, then $hg = 0$. g is epic $\Rightarrow h = 0$.

② $\tilde f \tilde g = 0$. For any $h \in \operatorname{Hom}_R(C, N)$, $\tilde f \tilde g h = hgf = h \circ 0 = 0$

③ $\operatorname{Ker}\tilde f \subseteq \operatorname{Im}\tilde g$. For $h \in \operatorname{Ker}\tilde f$, $\tilde f(h) = 0 = hf = 0$. Thus $h(\operatorname{Im} f) = 0$. From exactness of $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, $h(\operatorname{Ker} g) = 0$. This implies $\operatorname{Ker} g \subseteq \operatorname{Ker} h$. Then we need to use Prop 2.4:

$$\begin{tikzcd} B \arrow[r, two heads, "g"] \arrow[dr, "h"'] & C \arrow[d, dashed, "\exists h'"] \\ & N \end{tikzcd}$$

there exists a unique $h' \in \operatorname{Hom}_R(C, N)$ s.t. $h = h'g = \tilde g(h')$.

"$\Leftarrow$" ① $\operatorname{Im} g = C$. Take $N = C/\operatorname{Im} g$, we have $\operatorname{Ker}\tilde g = 0$. But quotient map $q: C \to N$ satisfies $\tilde g(q) = qg = 0$. Thus $q = 0$, $C/\operatorname{Im} g = 0$, $\operatorname{Im} g = C$.

② $\operatorname{Im} f \subseteq \operatorname{Ker} g$. Take $N = C$. $\tilde f \tilde g(\mathrm{id}_C) = 0$ implies $gf = 0$. Thus $\operatorname{Im} f \subseteq \operatorname{Ker} g$.

③ $\operatorname{Ker} g \subseteq \operatorname{Im} f$. Take $N = B/\operatorname{Im} f$. $q: B \to B/\operatorname{Im} f \in \operatorname{Hom}_R(B, N)$.

$q \in \ker \tilde f = \operatorname{Im}\tilde g$. Thus there is $h \in \operatorname{Hom}(C, N)$ such that $\tilde g(h) = q$.

$hg = q$, $\operatorname{Ker} q = \operatorname{Im} f$. For any $x \in \operatorname{Ker} g$, $hg(x) = 0 = q(x) \Rightarrow x \in \operatorname{Im} f$.

Prop 5.3 The following statements are equivalent:

(1) $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split exact.

(2) For any R module M

$$0 \longrightarrow Hom_R(M, A) \xrightarrow{\bar{f}} Hom_R(M, B) \xrightarrow{\bar{g}} Hom_R(M, C) \longrightarrow 0$$

is split exact.

(3) For any R module N

$$0 \longrightarrow Hom_R(C, N) \xrightarrow{\tilde{g}} Hom_R(B, N) \xrightarrow{\tilde{f}} Hom_R(A, N) \longrightarrow 0$$

is split exact.

Proof. (1) $\Rightarrow$ (2): We need to show the sequence is exact and split.

① prop 5.1 guarantees the exactness of

$$0 \longrightarrow Hom_R(M, A) \xrightarrow{\bar{f}} Hom_R(M, B) \xrightarrow{\bar{g}} Hom_R(M, C).$$

We need to show that $\bar{g}$ is epic. Since g has a section s such that $gs = id_C$. For any $h \in Hom_R(M, C)$ we define $h' \in Hom_R(M, B)$ as $h'(m) := sh(m)$. Then $g(h'(m)) = h(m)$. This means that $\bar{g}$ is surjective.

② To show the sequence is split, we need to show $\bar{g}$ is split. Since g is split, it has section $s: C \to B$, s.t. $gs = id_C$. This implies $\bar{g}\bar{s} = id$.

(2) $\Rightarrow$ (1): We need to show the exactness and splitness.

Prop 5.1 guarantees the exactness of

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C.$$

To show g is epic, take $M = C$, consider the exactness of

$$Hom_R(C, B) \xrightarrow{\bar{g}} Hom_R(C, C) \longrightarrow 0$$

$\bar{g}$ is epic. For $id_C \in Hom_R(C, C)$, there must exist $s \in Hom_R(C, B)$ such that $\bar{g}(s) = gs = id$. This implies g is split epic.

(1) $\Rightarrow$ (3). Exactness + splitness

Prop 5.2 guarantees $0 \longrightarrow Hom_R(C, N) \xrightarrow{\tilde{g}} Hom_R(B, N) \xrightarrow{\tilde{f}} Hom(A, N)$ is exact. To show it is split exact, we only need to show $\tilde{f}$ has a section.

Since f is split monic, it has a retraction $hf = id$. Consider

$$\tilde{h}: Hom_R(A, N) \longrightarrow Hom_R(B, N).$$

$\tilde{f}\circ\tilde{h} = \mathrm{id}$.

(3) $\Longrightarrow$ (1). Exactness + Splitness

We only need to show f has a retraction.

(II) Projective module.

Recall that for free module V and a epimorphism $f: M \to N$, we have commutative diagram:

$$\begin{tikzcd} M \arrow[r, two heads, "f"] & N \arrow[r] & 0 \\ V \arrow[u, dashed, "\exists h'"] \arrow[ur, "h"'] & & \end{tikzcd}$$

This means $\bar{f}: \mathrm{Hom}_R(V, M) \longrightarrow \mathrm{Hom}_R(V, N) \to 0$ is exact.

Does all module satisfy this property? No! But projective module has this property.

- Projective module $A \in {}_R\mathrm{Mod} \Leftrightarrow \mathrm{Hom}_R(A, \cdot)$ is exact.
- Injective module $A \in {}_R\mathrm{Mod} \Leftrightarrow \mathrm{Hom}_R(\cdot, A)$ is exact.
- Flat module $A \in \mathrm{Mod}_R \Leftrightarrow A \otimes_R \cdot$ is exact.

Def. 5.1. Let P be a R module, if for any epic $g: B \to C$ and module map $h: P \to C$, there exists $h': P \to B$ such that $h = gh'$, the P is called projective.

$$\begin{tikzcd} B \arrow[r, "g"] & C \arrow[r] & 0 \\ P \arrow[u, dashed, "\exists h'"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Thm. 5.4. All free modules are projective modules.

Proof. This is a result of prop 4.4.

Corollary. 5.5. Any R module is a quotient module of some projective module.

Thm 5.6. The following statements are equivalent:

(1) P is projective module.

(2) If $g: B \to C$ is epic, then $\bar{g}: \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$ is epic.

(3) If the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact, then

$$0 \longrightarrow \mathrm{Hom}_R(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}_R(P, C) \longrightarrow 0$$

is exact. Namely, $\mathrm{Hom}_R(P, \cdot)$ is exact functor.

(4) The short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$$

is split exact sequence.

(5) P is summand of some free module. Namely, there is a free module V and a submodule K of V such that $V = P \oplus K$.

Proof. (1) $\Leftrightarrow$ (2) is obvious from definition.

(2) $\Rightarrow$ (3): We only need to show $\bar{g}: \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, C)$ is epic, this is guaranteed by definition of projective module.

(3) $\Rightarrow$ (2): Consider exact sequence $0 \longrightarrow \ker g \xhookrightarrow{i} B \xrightarrow{g} C \longrightarrow 0$, we see $\bar{g}$ is epic.

(3) $\Rightarrow$ (4): Since $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$ is exact, we have exactness of

$$0 \longrightarrow \mathrm{Hom}_R(P, A) \xrightarrow{\bar{f}} \mathrm{Hom}_R(P, B) \xrightarrow{\bar{g}} \mathrm{Hom}_R(P, P) \longrightarrow 0.$$

Since $\bar{g}$ is epic, $\exists\, h \in \mathrm{Hom}_R(P, B)$ s.t. $\bar{g}(h) = id_P = gh$. Thus g is split epimorphism, the sequence is split exact (Splitting lemma).

(4) $\Rightarrow$ (5): Suppose P is quotient module of free module V, $g: V \twoheadrightarrow P$ is the quotient map. We obtain an exact sequence:

$$0 \longrightarrow \ker g \xhookrightarrow{i} V \xrightarrow{g} P \longrightarrow 0.$$

This is split exact, from splitting lemma $V \cong P \oplus \ker g$.

(5) $\Rightarrow$ (1): Suppose there is free module V and its submodule K such that

$$V = P \oplus K$$

Let $\pi: V \to P$ be canonical projection and $\iota: P \hookrightarrow V$ be canonical inclusion.

$$\begin{tikzcd}
 & V = P \oplus K \arrow[dl, "\exists h'"'] \arrow[d, shift right, "\pi"'] \\
 & P \arrow[u, shift right, "\iota"'] \arrow[d, "h"] \arrow[dl, dashed, "h''"'] \\
B \arrow[r, two heads, "g"'] & C \arrow[r] & 0
\end{tikzcd}$$

Since V is free, there exists a h' s.t. $gh' = h\pi$. Let $h'' = h'\iota$, it is clear that $gh'' = gh'\iota = h\pi\iota = h\circ id_P = h$.

<u>Exp</u> 5.1 Consider $R=V=\{\bar{0},\bar{1},\cdots,\bar{5}\}$, take its submodules $K=\{\bar{0},\bar{2},\bar{4}\}\cong\mathbb{Z}_3$ and $N=\{\bar{0},\bar{3}\}\cong\mathbb{Z}_2$, $V=K\oplus N$, since V is free, then K and N are projective.

<u>Prop</u> 5.7 Let $\{P_i\}_{i\in I}$ be a family of R modules, then $\bigoplus_{i\in I} P_i$ is projective R module iff every P_i is projective.

Proof. "$\Rightarrow$" obvious

"$\Leftarrow$" For projective modules $\{P_i\}_{i\in I}$ consider

$$\begin{tikzcd}
& P_j \arrow[d, "\iota_j"] \arrow[ddl, "\exists h_j"'] & \\
& \bigoplus_{j\in I} P_j \arrow[d, "h"] \arrow[dl, dashed, "\exists h'"'] & \\
B \arrow[r, two heads, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

To show that $gh'=h$, we need to use

$$\begin{tikzcd}
P_j \arrow[r, "\iota_j"] \arrow[dr, "gh_j"'] & \bigoplus_{j\in I} P_j \arrow[d, dashed, "\exists!\ g\bar{h}=h."] \\
& C
\end{tikzcd}$$

<u>Prop</u> 5.8 The projective module over PID is free module.

<u>Prop</u> 5.9 For projective module over PID, its submodule is projective.

<u>Prop</u>.* For PID D, finitely generated projective modules over $D[x_1,\cdots,x_n]$ are free.

(III) Dual module.

<u>Def</u>. 5.2 Let $M\in {}_R\mathrm{Mod}$, the dual module $M^* := \mathrm{Hom}_R(M,R)$. If $M\cong M^{**}$, M is called reflexive.

<u>Thm</u> 5.10. Let M be a R module.

(1) There exists a R module map $\theta: M\to M^{**}$.

(2) If M is free, θ is monic.

(3) If M is a free module and $\mathrm{rank}(M) < +\infty$, the θ is isomorphism.

Proof. (1) Define $\theta(a) : M^* \to R$ as $f \mapsto f(a)$.

(2) Suppose $a \in \ker\theta$ and $X = \{x_i\}_{i\in I}$ be basis of M, then $a = \sum r_i x_i$.

Define set map $\tilde{f}_i : X \to R$, $x \longmapsto \begin{cases} 1, & x = x_i \\ 0, & x \neq x_i \end{cases}$

we obtain a module map f_i by linear extension. Thus $f_i \in M^*$.

For any $f \in M^*$, we have $\theta(a)(f) = f(a) = \sum_i r_i f(x_i) = 0$.

Run over all $\{f_i\}_{i\in I}$, we obtain $r_i = 0$ for all $i \in I$.

(3) Since $\mathrm{rank}\, M = n < +\infty$. $X = \{x_1, \cdots, x_n\}$ is a basis of M. For any $f \in M^*$,

Let $s_i \in f(x_i)$ and $f' = \sum_i s_i f_i \in M^*$,

$$f(a) = \sum r_i f(x_i) = \sum_i r_i s_i = \sum_i s_i f_i (\sum_j r_j x_j).$$

This implies $\mathrm{span}\{f_1, \cdots, f_n\} = M^*$. $\{f_1, \cdots, f_n\}$ is also linearly independent.

Thus $\mathrm{rank}\, M^* = \mathrm{rank}\, M = n$, they are isomorphic.