Chapter 1 Module

§1.6 Injective Modules

- Duality
- Hom(• , M) and injective modules

(I) Duality

Duality: reversing the direction of arrows.

1. monomorphism f

$$\begin{tikzcd} K \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & M \arrow[r, "f"] & N \end{tikzcd}$$

1′ epimorphism f

$$\begin{tikzcd} K & M \arrow[l, shift right, "g"'] \arrow[l, shift left, "h"] & N \arrow[l, "f"'] \end{tikzcd}$$

2. product

$$\begin{tikzcd} \prod_{i\in I} M_i \arrow[r, "\pi_i"] & M_i \\ N \arrow[u, dashed, "\exists!\tilde{f}"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

2′ coproduct

$$\begin{tikzcd} \bigoplus_{i\in I} M_i \arrow[d, dashed, "\exists!\tilde{f}"'] & M_i \arrow[l, "l_i"'] \arrow[dl, "f_i"] \\ N & \end{tikzcd}$$

(II) Injective module

For R module I, $Hom(\bullet, I)$ is a contravariant functor:

$$M \longmapsto Hom(M, I)$$

$$(M \xrightarrow{f} N) \longmapsto Hom(f, I) = f^* \quad (\text{or denoted as } \tilde{f})$$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "f^*(h)"'] & N \arrow[d, "h"] \\ & I \end{tikzcd} \qquad f^*(h) = h\circ f$$

$$(f\circ g)^* = g^*\circ f^*, \quad (f\circ g)^*(h) = h\circ(f\circ g) = (h\circ f)\circ g = g^*\circ f^*(h).$$

Def 6.1 Let I be an R module, if for any monomorphism $f: A \to B$, and module map $h: A \to I$, there is module map $\bar{h}: B \to I$ such that $h = \bar{h}f$, then I is called an injective module.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "f"] \arrow[d, "h"'] & B \arrow[dl, dashed, "\exists \bar{h}"] \\ & I & \end{tikzcd}$$

Thm 6.1 The following statements are equivalent:

(1) I is injective module.

(2) If $f \in Hom(A, B)$ is monic, then $Hom(f, I) = \tilde{f} = f^*$ is epic.

(3) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is short exact sequence, then the following sequence is exact

$$0 \to Hom(C, I) \xrightarrow{g^*} Hom(B, I) \xrightarrow{f^*} Hom(A, I) \to 0.$$

(4) Exact sequence $0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split.

Proof. (1) $\Leftrightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Prop 5.2 guarantees the exactness of

$$0 \to Hom(C, I) \xrightarrow{g^*} Hom(B, I) \xrightarrow{f^*} Hom(A, I).$$

The only left part is to show f^* is epic, this is guaranteed by assumption (2).

(3) $\Rightarrow$ (4): Since $0 \to I \xrightarrow{f} B \xrightarrow{g} C \to 0$ exact, by assumption,

$$0 \to Hom(C, I) \xrightarrow{g^*} Hom(B, I) \xrightarrow{f^*} Hom(I, I) \to 0 \quad \text{exact}.$$

Since f^* is epic, there is h s.t. $f^*(h) = id_I$. Thus $hf = id_I$. f is split monic. From splitting lemma, the sequence is split exact.

(4) $\Rightarrow$ (1). To prove this, we need some subsequent result: ① any R module is a submodule of an injective module; ② $M \oplus N$ is injective iff M and N are injective. From ①, there is an injective E such that $I \hookrightarrow E$, we have the following exact sequence

$$0 \to I \overset{i}{\hookrightarrow} E \xrightarrow{q} E/I \to 0.$$

By assumption, it is split exact, thus $E \cong I \oplus E/I$. Now using ②, we see I is injective.

Prop 6.2′ For a finite family of R modules $\{M_j\}_{j\in J}$, $|J| < +\infty$, $\oplus_{j\in J} M_j$ is injective iff M_j are injective for all $j \in J$. Note that for infinite J, this does not hold.

Proof. ${}_R\mathrm{Mod}$ is additive category, then finite direct sum is isomorphic to finite direct product, we will prove a more general version in the next prop.

Prop 6.2. For index set J (not necessarily finite), direct product $\prod_{j\in J} M_j$ is injective iff M_j is injective for all $j \in J$.

Proof. "$\Leftarrow$" Suppose M_j's are injective, we need to show $\prod_{j\in J} M_j$ is injective.

Let $\pi_j : \prod_{j\in J} M_j \to M_j$ be canonical projections.

Consider monomorphism $f: A \hookrightarrow B$, $Hom(f, M_j)$ is epic for all $j \in J$.

For $g: A \to \prod_{j\in J} M_j$, define $g_j = \pi_j \circ g : A \to M_j$, the must exist $\tilde{g}_j : B \to M_j$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[dr, "g_j"'] & B \arrow[d, dashed, "\exists \tilde{g}_j"] \\ & & M_j \end{tikzcd} \qquad \pi_j g = \tilde{g}_j f$$

Then define $\tilde{g}: B \to \prod_{j\in J} M_j$ as $b \mapsto (\tilde{g}_j(b))_{j\in J}$, which is a R-module map and $\tilde{g} f(a) = (\tilde{g}_j(f(a)))_{j\in J} = (\pi_j g(a))_{j\in J} = g(a)$.

"$\Rightarrow$" We need to show

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h_j"'] & B \arrow[dl, dashed, "\exists \tilde{h}_j"] \\ & M_j & \end{tikzcd}$$

Define $h: A \to \prod_{j\in J} M_j$, $a \mapsto (h_j(a))_{j\in J}$, there exists $\tilde{h}$ s.t.

$$h = \tilde{h} \circ f$$

Define $\tilde{h}_j = \pi_j \tilde{h}$, we are done.

(III) Baer criterion and its application.

Every R module is a quotient module of a projective module

$$\Updownarrow \text{ dual}$$

Every R module is a submodule of an injective module.

Lemma 6.3 (Baer criterion). A left R module M is injective iff every R module map $f: I \to M$, where $I \leq R$ is an ideal, can be extended to R.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "f"'] & R \arrow[dl, dotted, "\exists \tilde{f}"] \\ & M & \end{tikzcd}$$

proof. "$\Rightarrow$": This is obvious, since inclusion $\iota: I \to R$ is monic.

"$\Leftarrow$": To show M is injective, consider

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \\ & M & \end{tikzcd}$$

We can simply regard $\mathrm{Im} f$ the same as A, i.e., using $a \in B$ to represent $f(a) \in B$. We need to extend h to B.

Let X be the set of pairs (A', h') wher $A \subseteq A' \subseteq B$ and $h': A' \to M$ extends h, meaning $h'|_A = h$. Define a partial order

$$(A', h') \preccurlyeq (A'', h'') \quad \text{iff} \quad A' \subseteq A'',\ h''|_{A'} = h'.$$

Using Zorn's lemma, there must exists a maximal element (A_0, h_0) in X.

If $A_0 = B$, we are done. Now assume $A_0 \neq B$.

There is $b \in B$ and $b \notin A_0$.

Define $I = \{r \in R : rb \in A_0\}$, I is an ideal of R.

Define $f: I \to M$ as $f(r) = h_0(r \cdot b)$, by assumption, this can extend to R as $\tilde{f}: R \to M$. Now, define $A_1 = A_0 + \langle b \rangle$, and $h_1: A_1 \to M$ as

$$h_1(a_0 + rb) := h_0(a_0) + r\tilde{f}(1).$$

This means $(A_1, h_1) \in X$ and $(A_1, h_1) \succcurlyeq (A_0, h_0)$, which leads a contradiction of maximality of (A_0, h_0).

Def 6.2 Let R be an integral domain, D is R module. If for any $y \in D$, and $0 \neq r \in R$, there is $x \in D$ such that $rx = y$, the D is called divisible.

remark. This means $0 \neq r \in R$, $r \triangleright (\bullet): D \to D$ defines a surjective map.

Example 6.1. $\mathbb{Q}$ is divisible as a $\mathbb{Z}$ module.

Example 6.2. $\mathbb{Q}/\mathbb{Z}$ is divisible as $\mathbb{Z}$ module.

Example 6.3. $\mathbb{Z}$ is not divisible as $\mathbb{Z}$ module.

Prop 6.4 Let R be an integral domain, then quotient of divisible module is divisible.

Prop 6.5 Let R be PID, then D is injective iff D is divisible.

Prop 6.6 Every abelian group can embed into a divisible abelian group (injective $\mathbb{Z}$ module).

For commutative ring R and abelian group A, $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is abelian group.

Define
$$R \times \mathrm{Hom}_{\mathbb{Z}}(R, A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, A)$$
$$(r, f) \longmapsto rf : a \mapsto f(a \cdot r)$$

then $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is a R module.

Lemma 6.7 If D is divisible abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective R module.

Theorem 6.8. Every R module can embed into an injective R module.

Proof. • Give $M \in {}_R\mathrm{Mod}$, it is an abelian group, thus $M \in {}_{\mathbb{Z}}\mathrm{Mod}$.

From prop 6.6, M can embed into a divisible abelian group D, $M \overset{f}{\hookrightarrow} D$.

Consider $0 \to M \overset{f}{\hookrightarrow} D$ in ${}_{\mathbb{Z}}\mathrm{Mod}$, $R \in {}_{\mathbb{Z}}\mathrm{Mod}$, we see $\mathrm{Hom}(R, f)$ is monic. $0 \to \mathrm{Hom}_{\mathbb{Z}}(R, M) \xrightarrow{f_*} \mathrm{Hom}_{\mathbb{Z}}(R, D)$.

• Since $\mathrm{Hom}_{\mathbb{Z}}(R, M)$, $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ are both R modules, we need to show f_* is an R module map, in this way, we obtain a R module embedding.

For $r, a \in R$, $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(R, M)$,
$$[f_*(r\alpha)](a) = (f \circ r\alpha)(a) = f(r \cdot \alpha(a)) = f(\alpha(ar)) = (f_*(\alpha))(ar)$$
$$= (r \cdot f_*)(a).$$

Thus f_* is R module map.

• Notice $\mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M)$. As R modules
$$\mathrm{Hom}_R(R, M) \leq \mathrm{Hom}_{\mathbb{Z}}(R, M).$$

But we know $M \cong \mathrm{Hom}_R(R, M)$, thus we have (in ${}_R\mathrm{Mod}$)
$$M \xrightarrow{\cong} \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \xrightarrow{f_*} \mathrm{Hom}_{\mathbb{Z}}(R, D).$$

Since D is divisible, prop 6.7 guarantees $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective.