Chapter 1 Module

§1.7 Tensor Product and Flat Modules

- Tensor product of modules
- Flat modules

(I) Tensor product of modules over commutative ring.

Def 7.1 Let A, B be R modules, the free module V is a free R module generated by elements in $A\times B$. Let K be a submodule of V generated by the following elements:

$$(a+a', b) - (a, b) - (a', b)$$

$$(a, b+b') - (a, b) - (a, b')$$

$$(ra, b) - r(a, b)$$

$$(a, rb) - r(a, b)$$

where, $r\in R$, $a, a'\in A$, $b, b'\in B$. The tensor product of A, B is defined as $A\otimes_R B = V/K$.

And we denote $a\otimes b := (a,b)+K$, $0 = 0\otimes 0$.

Exercise (1) Prove that $\bullet\otimes\bullet$ is bilinear.

(2) Prove that $0\otimes b = 0 = a\otimes 0$ $\forall a, b$.

(3) For modules, give examples to show that: there may exist $a\neq 0$, $b\neq 0$ but $a\otimes b=0$.

Hint: For $\mathbb{Z}$ module $\mathbb{Z}_m$ and $\mathbb{Z}_n$, $\mathbb{Z}_m\otimes_{\mathbb{Z}}\mathbb{Z}_N = \mathbb{Z}_{\gcd(m,n)}$. When $\gcd(n,m)=1$, $\mathbb{Z}_m\otimes_{\mathbb{Z}}\mathbb{Z}_n = 0$.

Def. 7.2. For R modules A, B, C, $f: A\times B\to C$ is called bilinear if

$$f(a+a', b) = f(a,b) + f(a', b)$$

$$f(ra, b) = r f(a, b)$$

$$f(a, b+b') = f(a,b) + f(a,b')$$

$$f(a, rb) = r f(a,b)$$

It is clear $\otimes: A\times B\to A\otimes_R B$ is bilinear.

Thm 7.1. Let A, B be R modules, for any R module C and bilinear map $f: A\times B\to C$, there exists a unique R module map $\bar{f}: A\otimes_R B\to C$ such that $\bar{f}\circ\otimes = f$. This is called universal property of $\otimes$.

$$\begin{tikzcd} A \times B \arrow[r, "f"] \arrow[d, "\otimes"] & C \\ A \otimes_R B \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

The tensor product is determined up to an isomorphism.

Proof. This is easy, try to prove it by yourself.

Prop 7.2. Let A, B be R modules, we have the following R module isomorphisms:

(1) $A \otimes_R R \cong A$, $\quad R \otimes_R B \cong B$

(2) $A \otimes_R B \cong B \otimes_R A$

Proof. (1) Define $f: A \times R \to A$, $(a, r) \mapsto r \cdot a$, it is clear that f is bilinear. By universal property, there is a linear map $\bar{f}: A \otimes_R R \to A$ such that

$$\bar{f}(a \otimes r) = f(a, r).$$

We can also define $g: A \to A \otimes_R R$, $a \mapsto a \otimes 1$. Since general element is $A \otimes_R R$ is of the form:

$$\textstyle\sum_i a_i \otimes r_i = (\sum_i r_i a_i) \otimes 1$$

Then $g \circ \bar{f}(a \otimes 1) = g(a) = a \otimes 1$, meaning $g \circ \bar{f} = id_{A \otimes_R R}$. And $\bar{f} \circ g = id_A$ is clear. Thus $A \otimes_R R \cong A$.

Similarly, we can prove $R \otimes_R B \cong B$.

(2) Define bilinear map $f: A \times B \to B \otimes_R A$, $(a, b) \mapsto b \otimes a$. This induce a R module map $\bar{f}: A \otimes_R B \to B \otimes_R A$ with $\bar{f}(a \otimes b) = b \otimes a$.

Similarly, we can obtain $\bar{g}: B \otimes_R A \to A \otimes_R B$ with $\bar{g}(b \otimes a) = a \otimes b$.

$\bar{f} \circ \bar{g} = id$, $\bar{g} \circ \bar{f} = id$.

Prop 7.3 Let A, B, C be R modules, then $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$.

Proof. Prove it by yourself.

Corollary 7.4. Let $f: A \to A'$, $g: B \to B'$ be R module maps, then there exists unique R module map $A \otimes_R B \longrightarrow A' \otimes_R B'$ s.t. for all $a \in A$, $b \in B$ we have

$$a \otimes b \longmapsto f(a) \otimes g(b).$$

We denote this map as $f \otimes g$.

Proof. Consider bilinear map $h: A \times B \longrightarrow A' \otimes_R B'$

$$(a, b) \longmapsto f(a) \otimes g(b),$$

it induces $f \otimes g$.

Remark. (1) $(f \otimes g) \circ (h \otimes l) = (f \circ h) \otimes (g \circ l)$.

(2) $(f \otimes g)^{-1} = f^{-1} \otimes g^{-1}$.

Prop 7.5 Let A, B, A_i, B_i be R modules, then

(1) $(\bigoplus_{i \in I} A_i) \otimes B \cong \bigoplus_{i \in I} (A_i \otimes B)$

(2) $A \otimes_R (\bigoplus_{i \in I} B_i) \cong \bigoplus_{i \in I} (A \otimes B_i)$

Proof. Exercise.

Prop. 7.6 Let A, B, C be R modules, then

$$\phi: \mathrm{Hom}_R(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}(A, \mathrm{Hom}_R(B, C))$$

$$f \longmapsto \phi(f): a \mapsto (b \mapsto f(a \otimes b))$$

is a R module isomorphism. ($\bullet \otimes_R B$ and $\mathrm{Hom}_R(B, \bullet)$ are adjoints !!)

Proof. Step 1. ϕ is well-defined. define $F_a \in \mathrm{Hom}_R(B, C)$

We need to show that $\phi(f): a \mapsto (b \mapsto f(a \otimes b))$ is a R module map.

Notice $\phi(f)(a + a') = F_{a+a'}$. $F_{a+a'}(b) = f((a+a') \otimes b) = f(a \otimes b) + f(a' \otimes b)$

$$= F_a(b) + F_{a'}(b)$$

Similarly $F_{ra} = r F_a$.

Step 2. ϕ is R module map.

$\phi(f + f')(a) = [\phi(f) + \phi(f')](a)$

acts both sides on b.

Similarly $\phi(rf) = r\phi(f)$.

Step 3. Define $\psi: \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) \longrightarrow \mathrm{Hom}(A \otimes_R B, C)$

$$g \longmapsto \psi(g): a \otimes b \mapsto (g(a))(b).$$

Repeat step 1 & 2 to show ψ is well-defined and a R module map.

Step 4. $\phi \circ \psi = \mathrm{id}$, $\psi \circ \phi = \mathrm{id}$.

Prop 7.7 Let V be a free R module, X is basis of V, then any element in $A \otimes_R V$ can be uniquely expressed as

$$\sum_{i=1}^{n} a_i \otimes x_i$$

where $a_i \in A$ and $x_i \in X$, $n \in \mathbb{Z}_+$.

Proof. Easy, try it.

Prop 7.8 Suppose A, B are free R modules with bases X and Y respectively, then $A \otimes_R B$ is also free and its basis is $\{x \otimes y \mid x \in X, y \in Y\}$.

Proof. Obvious.

(II) Flat module

Given a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R modules, the sequence

$$0 \to M \otimes_R A \xrightarrow{id \otimes f} M \otimes_R B \xrightarrow{id \otimes g} M \otimes_R C \to 0$$

is in general not exact.

Prop. 7.9 Let M be a R module, given exact seq

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

we have the following exact sequence:

$$M \otimes_R A \xrightarrow{id \otimes f} M \otimes_R B \xrightarrow{id \otimes g} M \otimes C \longrightarrow 0$$

$$A \otimes_R M \xrightarrow{f \otimes id} B \otimes_R M \xrightarrow{g \otimes id} M \otimes C \longrightarrow 0$$

Namely, $\bullet \otimes_R M$ and $M \otimes_R \bullet$ are right exact functor.

Prop 7.9': $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff for any M $A \otimes M \xrightarrow{f \otimes id} B \otimes M \xrightarrow{g \otimes id} C \otimes M \to 0$ is exact.

(since we can take $M = R$ and $N \otimes_R R \cong N$ for any N)

Proof. Step 1. $id \otimes C$ is epic. Obvious.

Step 2. $\operatorname{Im}(id \otimes f) \subseteq \operatorname{Ker}(id \otimes g)$.

$$(id \otimes g) \circ (id \otimes f) = id \otimes g f = id \otimes 0 = 0$$

Step 3. $\operatorname{Ker}(id \otimes g) \subseteq \operatorname{Im}(id \otimes f)$

This is elementary, try to prove it by yourself.

Proof'. Prop 5.1 and Prop 5.2

$0 \to A \xrightarrow{f} B \xrightarrow{g} C$ exact iff $\forall N$, $0 \to Hom(N, A) \xrightarrow{f_*} Hom(N, B) \xrightarrow{g_*} Hom(N, C)$ exact.

$A \to B \to C \to 0$ exact iff $\forall N$, $0 \to Hom(C,N) \xrightarrow{f^*} Hom(B,N) \xrightarrow{g^*} Hom(A,N)$ exact.

Now for any R module P set $N = Hom(M,P)$, we see that

$$0 \to Hom(C, Hom(M,P)) \xrightarrow{g^*} Hom(B, Hom(M,P)) \xrightarrow{f^*} Hom(A, Hom(M,P))$$

is exact.

From Prop 7.6, $Hom(X \otimes Y, Z) \cong Hom(X, Hom(Y,Z))$, we see

$$0 \to Hom(C \otimes M, P) \xrightarrow{\tilde{g^*}} Hom(B \otimes M, P) \xrightarrow{\tilde{f^*}} Hom(A \otimes M, P)$$

is exact. Since P is arbitrary, we see

$$A \otimes M \xrightarrow{f \otimes id} B \otimes M \xrightarrow{g \otimes id} C \otimes M \to 0$$

is exact.

You need to check $(g \otimes id)^* = \tilde{g^*}$ and $(f \otimes id)^* = \tilde{f^*}$!!

Remark $Hom(M, \cdot)$ is left exact.

$\cdot \otimes M$ is right exact, in general it is not exact.

Def. 7.2. Let M be a R module, if for any momorphism $f: A \to B$, the induce $id_M \otimes f : M \otimes_R A \to M \otimes_R B$ is also monomorphism, the M is ed flat module.

Prop. 7.10 M is flat module iff for any exact segu

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

the sequence

$$0 \to M \otimes A \xrightarrow{id \otimes f} M \otimes B \xrightarrow{id \otimes g} M \otimes C \to 0$$

is exact.

Proof. A direct result of Prop 7.9.

Prop 7.11 Let $\{M_i\}_{i \in I}$ be a family of R modules, $M = \oplus_{i \in I} M_i$ is flat iff every M_i is flat.

Proof. We need to show that for any monic map $f: A \to B$, the map

$$id_M \otimes f$$

is monic.

Since $M \otimes_R A = (\oplus_{i \in I} M_i) \otimes_R A \cong \oplus_{i \in I} (M_i \otimes_R A)$

$$M\otimes_R B = (\oplus_{i\in I} M_i)\otimes_R B \cong \oplus_{i\in I}(M_i\otimes_R B).$$

From the property of direct sum, $id_M\otimes f$ is monic iff $id_{M_i}\otimes f$ is monic.

Exercise. Given a family of modules $\{M_i\}_{i\in I}$, the direct sum $\oplus_{i\in I} M_i$ together with canonical embedding $l_i: M_i \hookrightarrow M$ has the property that: For module N and $f: \oplus_{i\in I} M_i \to N$, f is monic iff $f_i := f\circ l_i$ are monic for all $i\in I$.

Proof. "$\Rightarrow$": Since $f: \oplus_{i\in I} M_i \to N$ is monic and $l_i: M_i \to \oplus_{i\in I} M_i$ are monic, thus $f_i = f\circ l_i$ are also monic.

"$\Leftarrow$": Since $f_i = f\circ l_i$ are monic. For $x = (x_i)_{i\in I} \in \oplus_{i\in I} M_i$, $f(x)=0$ implies that $f(\sum_{j=1}^{k} l_{i_j}(x_{i_j})) = \sum_{j=1}^{k} f_{i_j}(x_{i_j}) = 0$. Thus $x=0$.

Prop 7.12 All projective modules are flat.

Proof. Step 1. $R\in {}_R\mathrm{Mod}$ is flat

Step 2. Free modules are flat!!

Since free module $V\cong \oplus_x V_x$ $V_x\cong R$

Prop 7.11 and step 1 guarantees the conc $\cdot$ n.

Step 3. Projective module P is a direct summand of free module V.

$V\cong P\oplus K$. Prop 7.1 guarantees the result.

Prop. 7.13. R module M is flat iff for all ideal $S\trianglelefteq R$, the following sequence

$$0\longrightarrow M\otimes_R S \xrightarrow{id_M\otimes i} M\otimes_R R$$

is exact, $i: S\hookrightarrow R$ is canonical embedding.

Proof. Omitted here.

Prop 7.14 R module M is flat iff: for any linear expression

$$\sum_{i=1}^{m} r_i x_i = 0, \quad r_i\in R,\ m_i\in M$$

there must exist $n\in \mathbb{Z}_+$ and $s_{ij}\in R$, $y_j\in M$, $i=1,\cdots,m$, $j=1,\cdots,n$, such that

$$x_i = \sum_{j=1}^{n} s_{ij} y_j, \quad i=1,\cdots,m$$

$$\sum_{i=1}^{m} r_i s_{ij} = 0, \quad j=1,\cdots,n.$$

Proof. Omitted here.

Prop 7.15 For PID R, R module M is flat iff M is torsion-free.

Proof. "$\Rightarrow$": When M is flat, if M is not torsion-free, then there exist $0 \neq a \in R$ and $0 \neq m \in M$ s.t. $a \cdot m = 0$.

Consider monic $f: R \to R$, $r \mapsto a \cdot r$ (monic since R is PID),

$f \otimes id_M : R \otimes_R M \longrightarrow R \otimes_R M$ is monic since M is flat.

But $(f \otimes id_M)(1 \otimes x) = a \otimes x = 1 \otimes ax = 1 \otimes 0 = 0$, thus we have a contradiction.

"$\Leftarrow$": Since R is PID, all ideals are of form $(a) = Ra$.

Let $a \neq 0$, $f: (a) \hookrightarrow R$ is embedding. Consided

$$f \otimes id_M : (a) \otimes_R M \longrightarrow R \otimes_R M.$$

Suppose $\sum_{i=1}^{m} r_i a \otimes x_i \in \ker f \otimes id_M$, then

$$\sum_{i=1}^{n} r_i a \otimes x_i = 1 \otimes a \sum_{i=1}^{n} r_i x_i = 0 \quad (\text{in } R \otimes_R M).$$

Thus $a \sum_{i=1}^{n} r_i x_i = 0$. Since $a \neq 0$ and M is torsion-free, thus

$$\sum_{i=1}^{n} r_i x_i = 0$$

This implies that

$$\sum_{i=1}^{n} r_i a \otimes x_i = \sum_{i=1}^{n} a \otimes r_i x_i = a \otimes \sum_{i=1}^{n} r_i x_i = a \otimes 0 = 0$$

in $(a) \otimes_R M$. Thus $f \otimes id_M$ is monic.

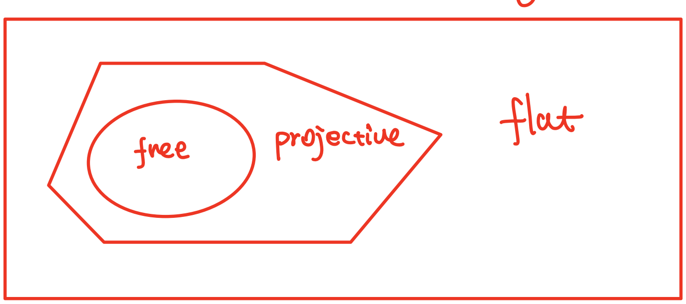
From Prop 7.13, M is flat.

Example 7.3. $\mathbb{Q}$ is torsion-free as $\mathbb{Z}$ module, thus $\mathbb{Q}$ is flat $\mathbb{Z}$ module.

But $\mathbb{Q}$ is not free $\mathbb{Z}$-module.

Summary

① For general unital commutative ring R



injective $\xleftrightarrow{\text{dual}}$ projective

② On PID R.

$$\text{free} \Leftrightarrow \underbrace{\text{projective} \Leftrightarrow \text{flat}}_{\text{holds for Noetherian ring !!}}$$