Chapter 1 Module

§1.8 Tensor Algebra, symmetric algebra, and exterior algebra

- Algebra
- Tensor algebra
- Symmetric algebra
- Exterior algebra

(I) Algebra

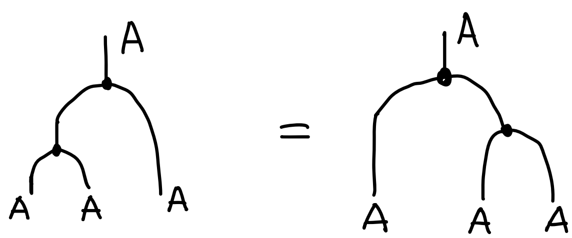
Def. 8.1 Let R be a commutative ring, A is a ring, A is called an algebra if

(1) $(A, +)$ is a R module.

(2) For any $r \in R$, $a, b \in A$, we have $\underline{r(ab) = (ra)b = a(rb)}$.

compatibility of ring structure and R module action.

Remark. $\mu: A \otimes_R A \to A$ satisfy



associativity axiom.

① $\mu(a \otimes b) = a \cdot b \Rightarrow \mu(ra \otimes b) = (ra) \cdot b$

$= \mu(a \otimes rb) = a \cdot (rb)$

② $A \in {}_R\mathrm{Mod} \Rightarrow A$ is an R module.

To show A is a ring:

- $(A, +)$ is abelian group is a result of $A \in {}_R\mathrm{Mod}$.
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ is a result of associativity axiom.
- $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a+b) \cdot c = a \cdot c + b \cdot c$ is from $\mu: A \otimes_R A \to A$ def.

Def. 8.1′ (1) If A has an identity element 1, A is called unital algebra.

(2) If A is commutative as a ring, it is called commutative algebra.

(3) If A is a divisible ring, A is called divisible algebra.

Example 8.1. Ring R is a $\mathbb{Z}$ algebra.

Example 8.2. $R[x_1, \cdots, x_n]$ and $R[[x_1, \cdots, x_n]]$ are R commutative algebra.

Example 8.3 The matrix ring $M_n(R)$ is a R algebra.

Example 8.4 Let A be a ring, R is a subring of Center(A), then A is a R algebra.

(II) Tensor algebra.

Let M be a R module. For integer $r \geq 0$, set

$$T^r(M) = \otimes_{i=1}^r M, \quad T^0(M) = R.$$

For $a_1 \otimes \cdots \otimes a_r \in T^r(M)$ and $b_1 \otimes \cdots \otimes b_s \in T^s(M)$, we can define their tensor product $(a_1 \otimes \cdots \otimes a_r) \otimes (b_1 \otimes \cdots \otimes b_s) \in T^{r+s}(M)$.

This gives a bilinear map $T^r(M) \times T^s(M) \to T^{r+s}(M)$, and this map is associative.

Def. The tensor algebra is defined as

$$T(M) := \oplus_{r=0}^{+\infty} T^r(M).$$

① It's clear that $T(M)$ is a R module.

② The ring structure of $T(M)$ is given by tensor product.

③ $r(a \bullet b) = (ra) \bullet b = a \bullet (rb)$.

Remark For free module M with rank $M = n$, $T^r(M)$ is also free and rank $T^r(M) = n^r$. And $T(M)$ is free module with rank $T(M) = +\infty$.

(III) Symmetric algebra.

For R module M, consider $T^r(M)$ and symmetry group S_r, define a submodule K_d^n generated by elements

$$a_1 \otimes \cdots \otimes a_r - a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(r)}.$$

Then we obtain a quotient module

$$S^r(M) := T^r(M) / K_d^n.$$

For elements in $S^r(M)$ we have

$$[a_1\otimes\cdots\otimes a_r] = [a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(r)}],$$

we call $[a_1\otimes\cdots\otimes a_r]$ symmetric product of $a_1,\cdots,a_r$.

Prop Define $K_S = \oplus_{r=0}^{+\infty} K_S^r$ and

$$S(M) := \oplus_{r=0}^{+\infty} S^r(M)$$

we have $S(M) \cong T(M)/K_S$.

Treate $T(M)$ as an algebra, K_S is an ideal of $T(M)$, thus $S(M)$ is the quotient algebra, called symmetric algebra.

Remark. For free module M with rank $M=n$, $x\in T^r(M)$, define

$$\mathrm{Sym}_r(x) = \frac{1}{r!}\sum_{\sigma\in S_n} \sigma x.$$

$S^r(M)$ has a basis $\{[e_{j_1}\otimes\cdots\otimes e_{j_r}] \mid 1\le j_1\le j_2\le\cdots\le j_r\le n\}$.

Thus $\mathrm{rank}\, S^r(M) = C_{n+r-1}^r$.

Remark. For free module M with rank $M=n$ and basis

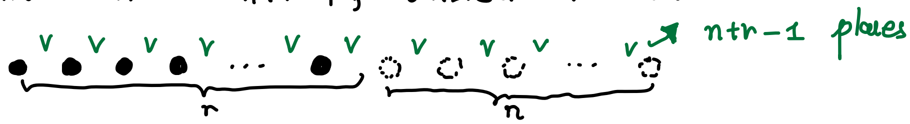
$$e_1,\cdots,e_n$$

we have basis of $S^r(M)$ as

$$[e_1^{\otimes\alpha_1}\otimes\cdots\otimes e_n^{\otimes\alpha_n}]$$

where $\alpha_1,\cdots,\alpha_n=0,\cdots,r$ and $\alpha_1+\cdots+\alpha_n=r$.

To see $\mathrm{rank}\, S^r(M) = C_{n+r-1}^r$, consider r balls



divide them into n groups by inserting $n-1$ dividers

$$C_{n+r-1}^{n-1} = C_{n+r-1}^r.$$

Prop. We have an algebra homomorphism for free module M with rank $M=n$:

$$f: R[x_1,\cdots,x_n] \longrightarrow S(M)$$

$$\sum a_{k_1\cdots k_n} x_1^{k_1}\cdots x_n^{k_n} \mapsto \sum a_{k_1\cdots k_n}[e_1^{\otimes k_1}\otimes\cdots\otimes e_n^{\otimes k_n}].$$

(IV) Exterior algebra

Exterior algebra is ubiquitous in geometry, it plays crucial role in constructing De Rham cohomology.

Def. For $T^r(M)$, define K_E^r as submodule generated by elements

$$a_1 \otimes \cdots \otimes a_r$$

where there exist $i \neq j$ s.t. $a_i = a_j$.

Define the quotient module as

$$\Lambda^r(M) := T^r(M) / K_E^r$$

its element is denoted as exterior product

$$a_1 \wedge \cdots \wedge a_r := [a_1 \otimes \cdots \otimes a_r].$$

Remark. ① $a_1 \wedge \cdots \wedge a_r = 0$ if $\exists\, i \neq j$ $a_i = a_j$

② $a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_j \wedge \cdots \wedge a_r = (-1)\, a_1 \wedge \cdots \wedge a_j \wedge \cdots \wedge a_i \wedge \cdots \wedge a_n.$

Proof. $a_1 \wedge \cdots \wedge \underbrace{(a_i + a_j)}_{i} \wedge \cdots \wedge \underbrace{a_i + a_j}_{j} \wedge \cdots \wedge a_r = 0$

Expand the expression, we obtain the expected result.

③ $a_{\sigma(1)} \wedge \cdots \wedge a_{\sigma(r)} = (-1)^{\operatorname{sgn} \sigma}\, a_1 \wedge \cdots \wedge a_r.$

Def. Let $K_E = \bigoplus_{r=0}^{+\infty} K_E^r$, then we have

$$\Lambda(M) := \bigoplus_{r=0}^{+\infty} \Lambda^r(M) \cong T(M) / K_E.$$

This is called exterior algebra.

Remark. For free module M with $\operatorname{rank} M = n$, if $r > n$ we have

$$\Lambda^r(M) = 0.$$

Thus $\Lambda(M) = \bigoplus_{r=0}^{n} \Lambda^r(M)$. $\quad \operatorname{Rank} \Lambda(M) = 2^n$.

Proof. $\operatorname{Rank} \Lambda^r(M) = C_n^r$

$$\operatorname{Rank} \Lambda(M) = \sum_{r=0}^{n} C_n^r = 2^n$$

Remark. For $v_j = \sum_i a_{ij} e_i$ with e_i basis of M.

$$v_1 \wedge \cdots \wedge v_n = \left(\sum_{i_1=1}^{n} a_{i_1 1} e_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^{n} a_{i_n n} e_{i_n}\right)$$

$$= \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} a_{i_1} \cdots a_{i_n}\, e_{i_1} \wedge \cdots \wedge e_{i_n}$$

$$= \sum_{i_1=1}^{n} \cdots \sum_{i_n=1}^{n} (-1)^{\operatorname{sgn}\begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}} a_{i_1 1} \cdots a_{i_n n}\, e_1 \wedge \cdots \wedge e_n$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} e_1 \wedge \cdots \wedge e_n.$$

Prop 8.1. Let V', V, V'' be free R modules with rank n', n, n'', a short exact sequence

$$0 \longrightarrow V' \xrightarrow{\varphi} V \xrightarrow{\psi} V'' \longrightarrow 0$$

induces a natral isomorphism

$$\Lambda^{n'}(V') \otimes \Lambda^{n''}(V'') \xrightarrow{\cong} \Lambda^{n}(V).$$

Proof. Notice rank $\Lambda^{n'}(V') = \text{rank}\, \Lambda^{n}(V) = \text{rank}\, \Lambda^{n''}(V'') = 1$.

We define $h: \Lambda^{n'}(V') \times \Lambda^{n''}(V'') \longrightarrow \Lambda^{n}(V)$ by

$$h(v_1' \wedge \cdots \wedge v_{n'}',\ v_1'' \wedge \cdots \wedge v_{n''}'') := \varphi(v_1') \wedge \cdots \wedge \varphi(v_{n'}') \wedge \psi^{-1}(v_1'') \wedge \cdots \wedge \psi^{-1}(v_{n''}'').$$

To show that h is well-defined, just notice $\psi^{-1}(v)$ is not uniquen but they differ with an element in $\operatorname{Ker}\psi = \operatorname{Im}\varphi$. But $\Lambda^{r}(\varphi(V')) = 0$ for $r > n'$.

Thus h is single-valued, thus well-defined

It's clear that h is bilinear.

$$\begin{tikzcd}
\Lambda^{n'}(V') \times \Lambda^{n''}(V'') \arrow[r, "h"] \arrow[d, "\otimes"] & \Lambda^{n}(V) \\
\Lambda^{n'}(V') \otimes \Lambda^{n''}(V'') \arrow[ur, dashed, "\exists!\, \bar{h}"'] &
\end{tikzcd}$$

Since $\text{Rank}\, \Lambda^{n}(V) = \text{Rank}\, \Lambda^{n'}(V') \otimes \Lambda^{n''}(V'') = 1$. To show $\bar{h}$ is isomorphic, we only need to show $\bar{h}$ is surjective. (Exercise)

Prop 8.2 Let $V = V' \oplus V''$ be direct sum of two free modules with finite rank, then for any $m \in \mathbb{Z}_+$, we have

$$\Lambda^{m}(V) \cong \bigoplus_{r+s=m} \Lambda^{r}(V') \otimes \Lambda^{s}(V''),$$

from which we obtain algebra isomorphism

$$\Lambda(V) \cong \Lambda(V') \otimes \Lambda(V'').$$

Proof. • Let $e_1', \cdots, e_{n'}'$ be basis of V' and $e_1'', \cdots, e_{n''}''$ be basis of V'', then

$$e_1', \cdots, e_{n'}',\ e_1'', \cdots, e_{n''}''$$

is basis of $V = V' \oplus V''$.

• For $0 \le r \le m$ define

$$\varphi_r: \Lambda^{r}(V') \otimes \Lambda^{m-r}(V'') \longrightarrow \Lambda^{m}(V)$$

by $\varphi_r((v'_{i_1}\wedge\cdots\wedge v'_{i_r})\otimes(v''_{j_1}\wedge\cdots\wedge v''_{j_{m-r}}))=v'_{i_1}\wedge\cdots v'_{i_r}\wedge v''_{j_1}\wedge\cdots\wedge v''_{j_{m-r}}$.

φ_r is monic module map.

- Define $\varphi=\bigoplus_{r=0}^{m}\varphi_r:\ \bigoplus_{r=0}^{m}\Lambda^r(V')\otimes\Lambda^{m-r}(V'')\longrightarrow\Lambda^m(V)$, φ is monic.
- $\sum_{r=0}^{m}\operatorname{rank}\Lambda^r(V')\ \operatorname{rank}\Lambda^{m-r}(V'')=\sum_{r=0}^{m}C_{n'}^{r}\,C_{n''}^{m-r}=C_{n+n'}^{m}=\operatorname{rank}\Lambda^m(V)$.

  Thus φ is isomorphism.

  This implies $\Lambda(V')\otimes\Lambda(V'')\cong\Lambda(V)$.
- Check the isomorphis is an algebra isomorphism.