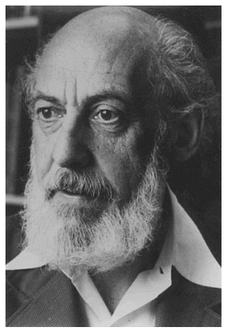
Chapter 2 Categories

§2.1 Definition of Categories

- Definition of categories.
- Examples of categories.

(I) Definition of categories

The notion of category was introduced by S. Eilenberg and S. MacLane in 1942.



Slogan: *Morphisms are the most crucial notion!*

Def 2.1. A category $\mathcal{C}$ consists of the following data:

(1) A class of objects $\mathrm{Ob}\,\mathcal{C}$.

(2) For any $A, B \in \mathrm{Ob}\,\mathcal{C}$, there is a set $\mathrm{Hom}(A, B)$, $f \in \mathrm{Hom}(A, B)$ is called morphism from A to B, denoted as $f: A \longrightarrow B$.

(3) For any triple A, B, C, the is a composition

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C)$$
$$(g, f) \longmapsto g \circ f.$$

They satisfy:

(i) $\mathrm{Hom}(A, B) = \mathrm{Hom}(A', B')$ iff $A = A'$ and $B = B'$

(ii) For any $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, we have

$$(h \circ g) \circ f = h \circ (g \circ f)$$

This means we can write down $f_1 \circ \cdots \circ f_n$ without ambiguity.

(iii) For any $A \in \mathrm{Ob}\,\mathcal{C}$, there is a special element $id_A \in \mathrm{Hom}(A, A)$, called identity, which satisfies $id_A \circ f = f$ and $g \circ id_A = g$ for all $f \in \mathrm{Hom}(B, A)$ and $g \in \mathrm{Hom}(A, B)$.

Def 2.2. • If $\mathrm{Ob}\,\mathcal{C}$ is a set, then $\mathcal{C}$ is called a small category.

• Note we assume $\mathrm{Hom}(A,B)$ be a set, this is not the case for general situatiation. This is category enriched in Set. If $\mathrm{Hom}(A,B)$ are sets for all A, B, the category is called locally small. In this sense, small category is category that is locally small and $\mathrm{Ob}\,\mathcal{C}$ is a set.

Def 2.3 For $f \in \mathrm{Hom}(A,B)$, if there is a $g \in \mathrm{Hom}(B,A)$ such that

$$f \circ g = id_B \text{ and } g \circ f = id_A$$

f is call isomorphism and $f^{-1} := g$. In this case, A and B are called isomorphic.

Def 2.4 For category $\mathcal{C}$, $\mathcal{D}$ is called a subcategory of $\mathcal{C}$ iff

(1) $\mathrm{Ob}\,\mathcal{D} \subseteq \mathrm{Ob}\,\mathcal{C}$

(2) $\mathrm{Hom}_{\mathcal{D}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$.

If for any A, B, $\mathrm{Hom}_{\mathcal{D}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)$, $\mathcal{D}$ is called a full subcategory of $\mathcal{C}$.

(II) Examples of categories.

Exp 1. The category of sets : Set

Exp 2. The group category : Grp

Exp 3. The Abelian group category: Ab

Exp 4. The ring category : Ring ; (Unital ring Ring_1)

Exp 5. The commutative ring category : CRing

Exp 6. The category of $\mathbb{F}$-vector spaces: $\mathrm{Vect}_{\mathbb{F}}$

Exp 7. The category of modules : ${}_R\mathrm{Mod}$, Mod_R, ${}_R\mathrm{Mod}_S$.

Exp 8. The category of topological spaces: Top

Exp 9. For patially-ordered set (poset), take $\mathrm{Ob}\,\mathcal{C} = P$, for any $a, b \in P$, define

$$\mathrm{Hom}(a,b) = \begin{cases} \{\cdot\}, & a \preccurlyeq b \\ \varnothing, & a \not\preccurlyeq b \end{cases}$$

$\{*\}$ means single point set.

This is a category.

Exp 10. Let $\mathrm{Ob}\,\mathcal{C} = \{*\}$, $\mathrm{Hom}(*,*) = G$ group G (or monoid G). $\mathcal{C}$ is a category.