Chapter 2 Categories

§2.2 Functor and Natural Transformation

- Functor
- Natural transformation

(I) Functor

$$\text{category } \mathcal{C} \xrightarrow{\text{functor } F} \text{category } \mathcal{D}$$

$$\text{object } A, B \longmapsto \text{objects } F(A), F(B)$$

$$\text{map } A \xrightarrow{f} B \longmapsto \text{map } F(A) \xrightarrow{F(f)} F(B)$$

Functors are morphisms in the category of all categories.

Def 2.1 For two categories $\mathcal{C}$ and $\mathcal{D}$, a covariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ consists of

(i) Maps on objects: $A \longrightarrow F(A)$

(ii) Maps on morphisms: $f \in Hom_{\mathcal{C}}(A, B) \longmapsto F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$.

They satisfy:

① $F(id_A) = id_{F(A)}$

② $F(g \circ f) = F(g) \circ F(f)$.

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "g\circ f"'] & & & F(A) \arrow[dr, "F(f)"] \arrow[dd, "F(g\circ f)"'] & \\
& B \arrow[dl, "g"] \arrow[r, squiggly, "F"] & {} & & F(B) \arrow[dl, "F(g)"] \\
C & & & F(C) &
\end{tikzcd}$$

Def. 2.2 $G: \mathcal{C} \longrightarrow \mathcal{D}$ is called contravariant functor if it maps $f \in Hom_{\mathcal{C}}(A, B)$ to $G(f) \in Hom_{\mathcal{D}}(G(B), G(A))$ such that

① $G(id_A) = id_{G(A)}$.

② $G(g \circ f) = G(f) \circ G(g)$.

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "g\circ f"'] & & & G(A) & \\
& B \arrow[dl, "g"] \arrow[r, rightsquigarrow] & {} & & G(B) \arrow[ul, "G(f)"'] \\
C & & & G(C) \arrow[uu, "G(g\circ f)"] \arrow[ur, "G(g)"'] &
\end{tikzcd}$$

Remark. (Duality). Dual category $\mathcal{C}^{op}$: ① $Ob\, \mathcal{C}^{op} = ob\, \mathcal{C}$ ② $Hom_{\mathcal{C}^{op}}(A, B) = Hom_{\mathcal{C}}(B, A)$.

$$(f\circ g)^{op} = g^{op} \circ f^{op}.$$

- A contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor $\widetilde{F}: \mathcal{C}^{op} \longrightarrow \mathcal{D}$.
- A contravariant functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a covariant functor $\widetilde{F}: \mathcal{C} \longrightarrow \mathcal{D}^{op}$.

For two functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{E}$, we can define their composition $G\circ F$. The ide functor can also be defined naturally.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[d, "id"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "F"'] & B \arrow[d, "F"] \\
F(A) \arrow[r, "F(f)"] \arrow[d, "G"'] & F(B) \arrow[d, "G"] \\
GF(A) \arrow[r, "GF(f)"] & GF(B)
\end{tikzcd}$$

Def 2.3. For functor $F: \mathcal{C} \to \mathcal{D}$, it is called:

(i) full, iff F is surjective on Hom set $F: Hom_{\mathcal{C}}(A, B) \twoheadrightarrow Hom_{\mathcal{D}}(F(A), F(B))$.

(ii) faithful, iff F is injective on Hom set $F: Hom_{\mathcal{C}}(A, B) \rightarrowtail Hom_{\mathcal{D}}(F(A), F(B))$.

Exp 2.1. Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$, we can define embedding by embedding of objects class and Hom set.

Exp 2.2 For group G, its commutator subgroup is defined as

$$[G, G] = \langle xyx^{-1}y^{-1} = [x, y] \mid x, y \in G \rangle.$$

The quotient group $G/[G, G]$ is called Abelianization of G.

This induces a functor $F: Grp \longrightarrow Ab$. (Check this !!)

This functor is left adjoint of inclusion functor $\iota: Ab \hookrightarrow Grp$.

Exp 2.3 For any set $X \in Set$. Define a free R module $\langle X \rangle$ with basis X. This gives a functor $F: Set \longrightarrow {}_R Mod$.

Similarly, we have $F: \mathrm{Set} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$. (Free functor)

Exp 2.4 Forgetful functor

$$\omega: {}_R\mathrm{Mod} \longrightarrow \mathrm{Set}$$

$$\omega: {}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}$$

(II) Hom functor and tensor functor

- $\mathrm{Hom}(\bullet, X): \mathcal{C} \longrightarrow \mathrm{Set}$ contravariant functor injective module
- $\mathrm{Hom}(X, \bullet): \mathcal{C} \longrightarrow \mathrm{Set}$ covariant functor projective module
- Tensor functor $A \otimes_R \bullet : {}_R\mathrm{Mod}_R \longrightarrow {}_R\mathrm{Mod}_R$ covariant functor flat module

(III) Natural transformation

Natural transformation is maps betweed functors.

Def 2.4. For two functors $F, G: \mathcal{C} \longrightarrow \mathcal{D}$, a natural transformation is a set of morphisms

$$\tau: F \longrightarrow G \quad = \{\tau_A \in \mathrm{Hom}_{\mathcal{D}}(F(A), G(A)) \mid A \in \mathcal{C}\}$$

such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\tau_A"'] & F(B) \arrow[d, "\tau_B"] \\
G(A) \arrow[r, "G(f)"] & G(B)
\end{tikzcd}$$

If all τ_A are isomorphisms, τ is called natural isomorphism.

Exp 2.5. For $\mathrm{Vect}_{\mathbb{F}}$, define doube-dual functor

$$F: \mathrm{Vect}_{\mathbb{F}} \longrightarrow \mathrm{Vect}_{\mathbb{F}}$$
$$V \longmapsto V^{\vee\vee}$$

Then $\{\theta_V : V \longrightarrow V^{\vee\vee}\}$ is a natural transformation

$$\theta: \mathrm{id}_{\mathrm{Vect}_{\mathbb{F}}} \longrightarrow F.$$

For $\mathrm{Vect}_{\mathbb{F}}^{FD}$, θ is natural isomorphism.