Chapter 2 Categories

2.4 representable functor and adjoint functor

- Representable functor
- Adjoint functor

(I) Representable functor

For any category $\mathcal{C}$, there are two functors from $\mathcal{C}$ to $\mathbf{Set}$

(1) Covariant $h_A = Hom_{\mathcal{C}}(A, \bullet)$

(2) Contravariant $h^B = Hom_{\mathcal{C}}(\bullet, B)$

Def. 4.1 Let $\mathcal{C}$ be a category, $F: \mathcal{C} \to \mathbf{Set}$ a covariant functor. If there is $A \in Ob\,\mathcal{C}$ and natural isomorphism

$$\alpha : Hom_{\mathcal{C}}(A, \bullet) \longrightarrow F$$

we say F is representable. The pair (A, α) is called representative of F.

Similarly, contravariant $F: \mathcal{C} \longrightarrow \mathbf{Set}$ is called representable iff there exists B such that $F \simeq h^B = Hom_{\mathcal{C}}(\bullet, B)$.

Example 4.1 Let $\mathcal{C}$ be a concrete category, and $X \in Ob\,\mathcal{C}$, define a Set valued functor $F: \mathcal{C} \longrightarrow \mathbf{Set}$ as follows

$$F(A) = Hom_{Set}(X, A).$$

$$F(f) = Hom_{Set}(X, f) = f_*$$

Suppose V is a free object over X, meaning there is $i: X \longrightarrow V$ such that for any $g: X \longrightarrow Y$, there is a unique $\tilde{g}$ such that $\tilde{g} \circ i = g$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "g"'] & V \arrow[d, dashed, "\exists!\, \tilde{g}"] \\ & Y \end{tikzcd}$$

F is representable with representative (V, α) where $\alpha: Hom(V, \bullet) \to F$ is defined as

$$\alpha_A : Hom_{\mathcal{C}}(V, A) \longrightarrow Hom_{set}(X, A)$$
$$\tilde{g} \longmapsto \tilde{g} \circ i$$

Example. 4.2. Let Mod_R be the category of R module, and $A, B \in Mod_R$. Define functor $F: Mod_R \longrightarrow Set$ as follows:

$$C \longmapsto Hom_{bilinear}(A\times B, C) = F(C)$$

$$\begin{array}{c} C \\ \downarrow g \\ D \end{array} \rightsquigarrow F(g) \begin{array}{c} F(C) \\ \downarrow \\ F(D) \end{array} \qquad A\times B \xrightarrow[bilinear]{f} C \xrightarrow{g} D$$

Then F is representable by $(A\otimes_R B, \alpha)$ with α defined as

$$\alpha: Hom_{Mod_R}(A\otimes_R B, \bullet) \longrightarrow F$$

$$Hom_{Mod_R}(A\otimes_R B, C) \xrightarrow{\alpha_C} F(C) = Hom_{bilinear}(A\times B, C)$$

$$h \xmapsto{\alpha_C} h\circ i$$

where $i: A\times B \longrightarrow A\otimes_R B$ is canonical map.

Lemma (Yoneda lemma). Let $F: \mathcal{C} \longrightarrow Set$ be a functor, then the natural transformations from F to $h_A = Hom_{\mathcal{C}}(A, \cdot)$ are in one-to-one correspondence with $F(A)$:

$$Nat(F, h_A) \cong F(A).$$

Moreover, this isomorphism is natural in A and F when both sides are regarded as functors from $Fun(\mathcal{C}, Set)\times \mathcal{C}$ to Set.

Example. $Nat(h_A, h_B) \cong h_A(B) = Hom_{\mathcal{C}}(A, B)$.

(II) Adjoint functor

Def. For two categories $\mathcal{C}$ and $\mathcal{D}$, the product category $\mathcal{C}\times\mathcal{D}$ is defined as

- $Ob\ \mathcal{C}\times\mathcal{D} = \{(X, Y) \mid X\in Ob\,\mathcal{C},\ Y\in Ob\,\mathcal{D}\}$
- $Hom((X, Y), (A, B)) = Hom(X, A)\times Hom(Y, B)$

$$\begin{tikzcd}
(X, Y) \arrow[r, "(f,g)"] \arrow[rr, bend right, "(k\circ f,\ l\circ g)"'] & (A, B) \arrow[r, "(k,l)"] & (C, D)
\end{tikzcd}$$

- We can consider $F: \mathcal{C} \times \mathcal{D} \longrightarrow \mathrm{Set}$, a typical example is $\mathrm{Hom}(\bullet, \bullet)$

covariant

contravariant

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

$$(X, Y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

$$\begin{tikzcd}
(X, Y) \arrow[d, "(f,g)"] \arrow[r, rightsquigarrow] & \mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f, g)"] & s \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[d, mapsto] \\
(X', Y') & \mathrm{Hom}_{\mathcal{C}}(X', Y') & g \circ s \circ f^{op}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "s"] & Y \arrow[d, "g"] \\
X' \arrow[u, "f^{op}"] \arrow[r] & Y'
\end{tikzcd}$$

- Consider covariant functors $F: \mathcal{C} \longrightarrow \mathcal{D}$, $G: \mathcal{D} \longrightarrow \mathcal{C}$. Then both of
  - $\mathrm{Hom}_{\mathcal{D}}(F(\bullet), \bullet)$
  - $\mathrm{Hom}_{\mathcal{C}}(\bullet, G(\bullet))$

  are functors from $\mathcal{C}^{op} \times \mathcal{D}$ to Set.

  A natural transformation $\alpha: \mathrm{Hom}_{\mathcal{C}}(\bullet, G(\bullet)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(\bullet), \bullet)$ is a set of maps

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

  such that for $f: X' \longrightarrow X$, $g: Y \longrightarrow Y'$, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \arrow[r, "\alpha_{X,Y}"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(f, G(g))"] & \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \arrow[d, "\mathrm{Hom}_{\mathcal{D}}(F(f), g)"] \\
\mathrm{Hom}_{\mathcal{C}}(X', G(Y')) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(F(X'), Y')
\end{tikzcd}$$

Def 4.2 Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ and $G: \mathcal{D} \longrightarrow \mathcal{C}$ be two functors. If there exists a natural isomorphism

$$\alpha: \mathrm{Hom}_{\mathcal{C}}(\cdot, G(\cdot)) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(\cdot), \cdot)$$

we say that F is left adjoint of G and G is right adjoint of F. Denote $F \dashv G$.

Example. In section 7 we prove that

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$$

This means $\bullet \otimes_R B \dashv \mathrm{Hom}(B, \bullet)$.

Prop 4.2 Funtor $G: \mathcal{D} \longrightarrow \mathcal{C}$ has left adjoint iff for any $C \in \mathrm{ob}\,\mathcal{C}$, hom-functor $\mathrm{Hom}(C, G(\cdot))$ is representable.

Proof. "$\Rightarrow$" Let $F \dashv G$, then there is natural isomorphism

$$\alpha_{C,D}: \mathrm{Hom}_{D}(F(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

Fix C, we see $\mathrm{Hom}_{D}(F(C), \cdot) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G(\cdot))$, meaning $\mathrm{Hom}(C, G(\cdot))$ is representable.

"$\Leftarrow$" Suppose (A_C, α) be representative of $\mathrm{Hom}_{\mathcal{C}}(C, G(\cdot))$

Define $F(C) = A_C$, we can check

$$\mathrm{Hom}(F(\cdot), \cdot) \xrightarrow{\cong} \mathrm{Hom}(\cdot, G(\cdot)).$$