Chapter 2 Categories

2.5 Abelian Categories

- monomorphisms and epimorphisms
- Kernel and cokernel; equalizer and coequalizer
- Additive category, Abelian category, additive functor

(I) Monomorphism and epimorphism

Def. 5.1 Let $f: C \longrightarrow D$ be a map in category $\mathcal{C}$.

(i) If for any $B \in \mathrm{Ob}\,\mathcal{C}$ and $g, h \in \mathrm{Hom}(B, C)$, $f\circ h = f\circ g \Rightarrow g = h$, then f is called monomorphism or monic map.

(ii) If for any $E \in \mathrm{Ob}\,\mathcal{C}$, and $u, v \in \mathrm{Hom}(D, E)$, $u\circ f = v\circ f \Rightarrow u = v$, then f is called epimorphism or epic map.

Example 1. In Set, Grp, Mod_R, ${}_R\mathrm{Mod}$:

monic = injective; epic = surjective.

Example 1. In Ring:

monic = injective; epic $\neq$ surjective

Surjective ring map is epic, but epic ring map is not necessarily surjective.

$f: \mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic but not surjective

Consider $R \in$ Ring, $u, v: \mathbb{Q} \to R$, if $uf = vf$ means $u(n) = v(n)$, $\forall n \in \mathbb{Z}$.

This implies that
$$\begin{aligned} u(\tfrac{1}{n}) &= u(\tfrac{1}{n}) \cdot v(1) \\ &= u(\tfrac{1}{n})\, v(n)\, v(\tfrac{1}{n}) \\ &= u(\tfrac{1}{n})\, u(n)\, v(\tfrac{1}{n}) \\ &= u(1) \cdot v(\tfrac{1}{n}) \\ &= v(\tfrac{1}{n}) \end{aligned}$$

Thus $u(\frac{m}{n}) = v(\frac{m}{n})$ $\forall \frac{m}{n} \in \mathbb{Q}$. f is epic.

Example 5.3 There exists monic map that is not injective map.

An Abelian group $(G, +)$ is called divisible if $\forall n \in \mathbb{Z}_+$ and $g \in G$, $\exists y \in G$ s.t. $ny = g$.

This is equivalent to: for any positive integer n, $nG = G$.

Canonical map $f: \underset{\text{divisible}}{\underline{\mathbb{Q}}} \longrightarrow \underset{\text{divisible}}{\underline{\mathbb{Q}/\mathbb{Z}}}$ is monic in divisible Abelian group category, but it is not injective. (every quotient group of divisible group is divisible)

For $A \in Ab^{div}$, $g, h: A \to \mathbb{Q}$ satisfy $fg = fh$. Then $\forall x \in A$, we have

$$f\,g(x) = f\,h(x) \quad \text{in } \mathbb{Q}/\mathbb{Z}$$

Thus $g(x) - h(x) \in \mathbb{Z}$ in $\mathbb{Q}$, If $g \neq h$, there exicts $x \in A$ s.t. $g(x) \neq h(x)$, and $g(x) - h(x) = n \neq 0$. Since A is divisible, $\exists\, y \in A$ s.t. $x = 2ny$.

Then $g(2ny) - h(2ny) = n \neq 0 \Rightarrow 2[g(y) - h(y)] = 1$ in $\mathbb{Q}$

$$\Rightarrow g(y) - h(y) = \frac{1}{2} \text{ in } \mathbb{Q}$$

This is in contradiction with assumption $g(x) - h(x) \in \mathbb{Z}$ in $\mathbb{Q}$ for all $x \in A$.

Thus $g = h$, f is monic.

Prop 5.1. Let $f: A \to B$, $g: B \to C$ be maps in $\mathcal{C}$.

(1) If f, g are monic, then $g \circ f$ is monic

(2) If gf is monic, then f is monic

(3) If f, g are epic, then gf is epic

(4) If gf is epic, then g is epic

(5) If f is isomorphism (meaning it has left and right inverses), then f is monic and epic, but the reverse direction is in general not true.

Proof. (1) Obvious

(2) Suppose $fu = fv$, then $gfu = gfv$, since gf monic, we see $u = v$

(3) Obvious

(4) Suppose $ug = vg$, then $ugf = vgf$, since gf epic, $u = v$.

(5) Left inverse $\Rightarrow$ left cancellation.

Right inverse $\Rightarrow$ right cancellation.

(II) Kernel and cokernel.

In category $\mathcal{C}$, zero object $0 \in \mathrm{Ob}\,\mathcal{C}$ is an object which initial and terminal.

Zero object, if exist, is unique up to isomorphism.

$$\#\,\mathrm{Hom}(0, A) = \#\,\mathrm{Hom}(A, 0) = 1.$$

Prop 5.2 Let $\mathcal{C}$ be a category that has zero object.

(1) $\forall A \in \mathrm{Ob}\,\mathcal{C}$, $0 \longrightarrow A$ is monic and $A \longrightarrow 0$ is epic

(2) $\forall B, C \in \mathrm{Ob}\,\mathcal{C}$, $\exists!\ 0_{CB} \in \mathrm{Hom}(B, C)$ called zero morphism, such that $\forall f \in \mathrm{Hom}(A, B)$, $\forall g \in \mathrm{Hom}(C, D)$, we have

$$0_{CB}\, f = 0_{CA}, \quad g\, 0_{CB} = 0_{DB}$$

Proof. (1) $f_A : 0 \longrightarrow A$. Since $\#\,\mathrm{Hom}(B, 0) = 1$. there is unique $u \in \mathrm{Hom}(B, 0)$ $f_A u : B \longrightarrow A$. $\Rightarrow$ f_A monic

Similary $g_A : A \to 0$ is epic.

(2) Existence. Define 0_{CB} as

$$B \longrightarrow 0 \longrightarrow C \;=\; B \xrightarrow{0_{CB}} C$$

$$A \xrightarrow{f} B \longrightarrow 0 \longrightarrow C \;=\; A \xrightarrow{0_{CA}} C$$

$$B \longrightarrow 0 \longrightarrow C \xrightarrow{g} D \;=\; B \xrightarrow{0_{DB}} D$$

Uniqueness. $\{0_{CB}\}_{C,B \in \mathrm{Ob}\,\mathcal{C}}$, $\{0'_{CB}\}_{CB}$ be different zero maps, then

$$0_{CA} = 0_{CB}\, 0'_{BA} = 0'_{CA}.$$

Def 5.2 For a category $\mathcal{C}$, let $f, g \in \mathrm{Hom}(A, B)$ be maps $A \overset{f}{\underset{g}{\rightrightarrows}} B$.

A fork consists of an object E and map $E \xrightarrow{i} A$ such that

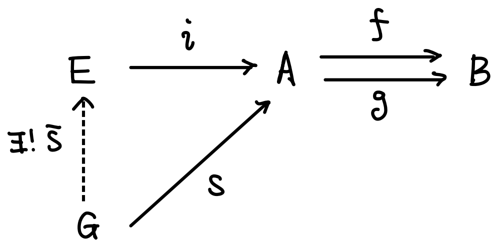
$f i = g i$ $\qquad E \xrightarrow{i} A \overset{f}{\underset{g}{\rightrightarrows}} B$.

An equalizer of f and g is an object E together with map $i : E \to A$ such that $E \xrightarrow{i} A \overset{f}{\underset{g}{\rightrightarrows}} B$ is a fork, and it satisfies the following universal property:

For any fork $G \xrightarrow{s} A \overset{f}{\underset{g}{\rightrightarrows}} B$, there exists a unique map

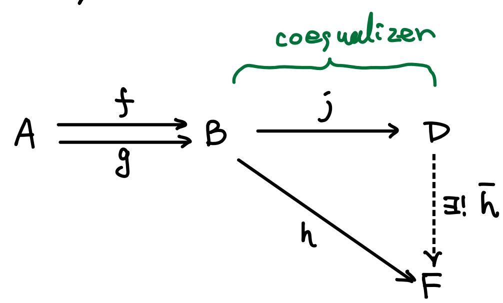
$$\bar{s}: G \longrightarrow E$$

such that the following diagram commute:



If $\mathcal{C}$ has zero objects, the equalizer of $(f, 0_{AB})$ is called kernel of f.

Def 5.2′. The coequalizer and cokernel are dual concepts of equalizer and kernel.



The coequalizer of $(f, 0_{AB})$ is called cokernel of f.

Example 5.4 In Grp, Ring, Mod_R equalizer of $f: A \to B$ and $g: A \to B$ is $K := \{x \in A \mid f(x) = g(x)\}$ equipped with embedding $i: K \hookrightarrow A$.

In Mod_R, coequalizer of f and g is $C = B/\text{Im}(f-g)$ equipped with quotient map $q: B \longrightarrow B/\text{Im}(f-g)$.

Prop. For $A \overset{f}{\underset{g}{\rightrightarrows}} B$, their equalizer map is monic

their coequalizer map is epic

Proof. (i) equalizer map is monic

consider $u: X \longrightarrow E$, $v: X \longrightarrow E$, we need to show that

$iu = iv$ implies $u = v$.

Set $t = iu = iv$, we see $ft = fiu = giu = gt$. Thus t equalizes f and g

$$\begin{array}{ccccc} X & \xrightarrow{t} & A & \overset{f}{\underset{g}{\rightrightarrows}} & B \\ {\scriptstyle \exists! \bar{t}} \downarrow & \nearrow_{i} & & & \\ E & & & & \end{array}$$

Notice $\bar{t}$ is unique, but we see ① set $u = \bar{t}$ or ② set $v = \bar{t}$, the diagram commutes. Thus we must have $u = v$.

(ii) Similar.

Prop Equalizer is terminal object $\mathcal{C}_{f,g}$

Coequalizer is initial object $\mathcal{D}_{f,g}$

Proof. Exercise.

(III) Abelian category and additive category.

Def 5.3 (Additive category) An additive category $\mathcal{C}$ is a category satisfies:

(1) $\mathcal{C}$ has zero object

(2) For any $A, B \in \operatorname{Ob} \mathcal{C}$, $\operatorname{Hom}(A, B)$ is an Abelian additive group with zero element 0_{AB}.

(3) Composition of morphisms is bilinear in the sense that

$$(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$$
$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$$

(4) For any finite $A_1, \cdots, A_n \in \operatorname{Ob} \mathcal{C}$, there is an object A which is simultaneously product and coproduct of $A_1, \cdots, A_n$. A is called direct sum and we denote $A = A_1 \oplus \cdots \oplus A_n$.

Def 5.4 (Abelian category) $\mathcal{C}$ is an Abelian category if it is additive category and it satisfies

(1) Every morphism has kernel and cokernel

(2) Every monomorphis is kernel of its cokernel, every epimorphism is cokernel of it kernel.

Remark. There are many equivalent definitions of Abelian category.

Example. Ab and Mod_R are Abelian categories, but Grp, Ring are not Abelian categories.

E.g., for groups A, B and monic $f: A \hookrightarrow B$, $Im f$ is a subgroup of B. To define cokernel $B/Im f$ $Im f$ must be normal subgroup of B, this is in general not the case.

Def 5.5 Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between two Abelian categorries, if for any $A, B \in Ob\, \mathcal{C}$ we have $F(A \oplus B) = F(A) \oplus F(B)$.

Prop 5.4 Let $F: \mathcal{C} \to \mathcal{D}$ be additive functor between Abelian categories, then F is a group homomorphism between $Hom(A, B)$ and $Hom(F(C), F(D))$

$$F(f+g) = F(f) + F(g), \quad F(0) = 0.$$

Moreover, additive functor maps split exact sequence to split exact sequence.

Def. 5.6. Consider additive $F: \mathcal{C} \longrightarrow \mathcal{D}$ between Abelian categories

- F is right exact if

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \quad \text{exact}$$

$$\Downarrow$$

$$F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \longrightarrow 0 \quad \text{exact}$$

- F is left exact if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \quad \text{exact}$$

$$\Downarrow F$$

$$0 \longrightarrow F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \quad \text{exact}$$

- F is exact if F is left and right exact

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \quad \text{exact}$$

$$\Downarrow$$

$$0 \longrightarrow F(M') \xrightarrow{F(f)} F(M) \xrightarrow{F(g)} F(M'') \longrightarrow 0 \quad \text{exact}$$

Def 5.6′ For contravariant additive functor $F: \mathcal{C} \longrightarrow \mathcal{D}$, left, right exactness can be defined similarly.

Theorem (Mitchell embedding) Every Abelian category $\mathcal{C}$ is equivalent, as additive category, to a full subcategory of ${}_R\mathrm{Mod}$ over some unital ring R.