Chapter 2 Categories

2.6 Limit and Colimit

- Direct and inverse limits for modules
- Limit
- Colimit

(I) Direct limit of modules

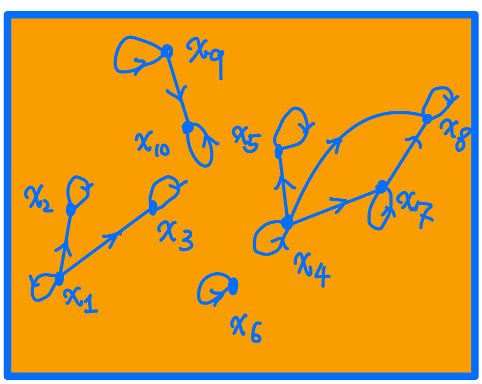
<u>Def</u> (Poset) A partially ordered set (poset) is a set I equipped with a partial order "$\precsim$":

1. $x \precsim x \quad \forall\, x \in I$.
2. If $x \precsim y$ and $y \precsim x$, then $x = y$.
3. If $x \precsim y$ and $y \precsim z$, then $x \precsim z$.

Exercise. Every poset $(I, \precsim)$ can be regarded as a category:

1. $\mathrm{Ob}\, \mathcal{I} = I$
2. $\mathrm{Hom}(x, y) = \{*\}$ if $x \precsim y$

   $\mathrm{Hom}(x, y) = \phi$, if $x \not\precsim y$



$$
\begin{tikzcd}
x \arrow[r, "\precsim_{xy}"] \arrow[rr, bend right, "\precsim_{xz}"'] & y \arrow[r, "\precsim_{yz}"] & z
\end{tikzcd}
$$

$\precsim_{yz} \circ \precsim_{xy} = \precsim_{xz}$

$\precsim_{xz}$ must exist

<u>Def</u>. A direct set I is a poset such that any finite subset has an upper bound.
More precise, I is a poset such that for any $x, y \in I$, there is a $z \in I$ such that $x \precsim z$ and $y \precsim z$.

Example. Every totally ordered set is direct set.

Counter example is discrete poset $\{x, y, z, \cdots\}$, where we only have $x \precsim x$, $y \precsim y$, $z \precsim z$, ...

Def Consider category of direct set I and category Mod_R of R-modules, a covariant functor

$$F: I \longrightarrow \mathrm{Mod}_R$$

gives a family of R-modules $\{M_\lambda\}_{\lambda \in I}$ and module maps $\{f_{\lambda\mu} \in \mathrm{Hom}(M_\mu, M_\lambda)\}$ that satisfy

1. For every $\lambda \in I$, $f_{\lambda\lambda} = \mathrm{id}_{M_\lambda}$
2. For any $\lambda \preccurlyeq \mu \preccurlyeq \nu$, we have $f_{\nu\mu} \circ f_{\mu\lambda} = f_{\nu\lambda}$

We call $\{M_\lambda, f_{\mu\lambda}\}$ a direct system (or, we call $F: I \to \mathrm{Mod}_R$ a direct system).

Def For a direct system $\{M_\lambda, f_{\mu\lambda}\}$, if there exists a module M and a set of R-module maps $\{\varphi_\lambda : M_\lambda \longrightarrow M\}$ such that

1. For any $\lambda \preccurlyeq \mu$, we have $\varphi_\lambda = \varphi_\mu \circ f_{\mu\lambda}$.
2. If there exist another $N \in \mathrm{Mod}_R$ and R-module maps $\{\psi_\lambda : M_\lambda \longrightarrow N\}$ such that for any $\lambda \preccurlyeq \mu$, we have $\psi_\lambda = \psi_\mu \circ f_{\mu\lambda}$, then there exists a unique $g: M \longrightarrow N$ such that $\psi_\lambda = g\varphi_\lambda \quad \forall \lambda \in I$.

$$\begin{tikzcd} M_\lambda \arrow[dr, "\varphi_\lambda"] \arrow[dd, "f_{\mu\lambda}"'] & \\ & M \\ M_\mu \arrow[ur, "\varphi_\mu"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M \arrow[dd, dashed, "\exists! g"] \\ M_\lambda \arrow[ur, "\varphi_\lambda"] \arrow[dr, "\psi_\lambda"'] & \\ & N \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} M \arrow[d, equal] & & \\ \varinjlim M_\lambda \arrow[rr, dashed, "\exists! g"] & & N \\ & M_\lambda \arrow[ul, "\varphi_\lambda"] \arrow[ur, "\psi_\lambda"'] \arrow[d, "f_{\mu\lambda}"] & \\ & M_\mu \arrow[uul, bend left, "\varphi_\mu"] \arrow[uur, bend right, "\psi_\mu"'] & \end{tikzcd}$$

Then M is called the direct limit of direct system $\{M_\lambda, f_{\lambda\mu}\}$, and we denote

$$\varinjlim M_\lambda = M.$$

Prop In Mod_R, the direct limit always exists.

Proof. Consider direct sum $\bigoplus_{\alpha \in I} M_\alpha$ with canonical map $\iota_\alpha : M_\alpha \longrightarrow M$. Construct submodule S as

$$S = \langle \iota_\mu \circ f_{\mu\lambda}(x_\lambda) - \iota_\lambda(x_\lambda) \mid \lambda \preccurlyeq \mu,\ x_\lambda \in M_\lambda \rangle$$

and we define $M = \bigoplus_{\alpha \in I} M_\alpha / S$, the quotient map is $q: \bigoplus_{\alpha \in I} M_\alpha \longrightarrow M$. Then we define $\varphi_\lambda = q \circ \iota_\lambda : m_\lambda \mapsto \iota_\lambda(m_\lambda) + S$, the $(M, \{\varphi_\lambda : M_\lambda \to M\})$ is the direct limit.

We need to check M satisfies universal property.

$$\begin{tikzcd}
M_\lambda \arrow[dr, "\iota_\lambda"] \arrow[dd, "f_{\mu\lambda}"'] \arrow[rrrd, bend left, "\psi_\lambda"] & & & \\
& \bigoplus_{\alpha\in I} M_\alpha \arrow[r, "q"] \arrow[rr, dashed, bend right, "\exists!\, \tilde{\psi} \text{ (dir sum)}"'] & M=\bigoplus_{\alpha\in I} M_\alpha/S \arrow[r, dashed, "\exists!\, g"] & N \\
M_\mu \arrow[ur, "\iota_\mu"'] \arrow[rrru, bend right, "\psi_\mu"'] & & &
\end{tikzcd}$$

- $S \subseteq \operatorname{Ker}\tilde{\psi}$, thus there exist a canonical $g: M \longrightarrow N$.

$$\underline{\psi_\lambda = \tilde{\psi}\circ \iota_\lambda = \tilde{\psi}\circ \iota_\mu \circ f_{\mu\lambda}} = \psi_\mu \circ f_{\mu\lambda}$$

$$\psi_\mu = \tilde{\psi}\circ \iota_\mu$$

$$S \subseteq \ker\tilde{\psi}$$

Define $g: \begin{array}{ccc} M & \longrightarrow & N \\ m+S & \longmapsto & \tilde{\psi}(m) \end{array}$, which is well-defined since $S \subseteq \operatorname{Ker}\psi$.

Such a map is unique, since any $h: M/S \longrightarrow N$ with $h\circ q = \tilde{\psi}$ must have $h(m+S) = \tilde{\psi}(m) = g(m+S)$.

Example 6.1. Consider a R-module M, $\{M_\lambda\}_{\lambda\in I}$ is the set of all finitely generated submodules. Set partial or of I as: $\lambda \preceq \mu$ iff $M_\lambda \subseteq M_\mu$. Since for any M_λ and M_μ, there exists a finitely generated $M_\lambda + M_\mu$ that contain M_λ, M_μ as submodules, thus I is a direct set.

When $M_\lambda \subseteq M_\mu$, define $f_{\mu\lambda}: M_\lambda \hookrightarrow M_\mu$ as inclusion, then $\{M_\lambda, f_{\mu\lambda}\}$ is a direct system. For this we have ($\varphi_\lambda: M_\lambda \longrightarrow M$ is inclusion map):

$$\varinjlim M_\lambda = M.$$

Namely, every module is a direct limit of its finitely generated submodules.

Proof. Try to prove it by yourself, you will need to use the fact that

$$M = \bigcup_{N\in \mathcal{F}_M} N$$

where $\mathcal{F}_M$ is the set of all finitely generated modules.

This is exercise 17 of Sec 2 of Atiyah & Macdonald

Prop For direct system $\{M_\lambda, f_{\lambda\mu}\}$ and R-module N:

$$\varinjlim M_\lambda \otimes_R N \cong (\varinjlim M_\lambda) \otimes_R N.$$

Proof. Exercise 20 of Sec 2 of Atiyah & Macdonald.

(II) Inverse limit of modules

Def. For direct index set I, consider contravariant functor

$$F : I \longrightarrow \mathrm{Mod}_R.$$

We have a family of modules $\{M_\lambda\}$ and R-module maps $\{f_{\lambda\mu} \in \mathrm{Hom}(M_\mu, M_\lambda)\}$ such that

1. For any $\lambda \in I$, $f_{\lambda\lambda} = id_{M_\lambda}$
2. For any $\lambda \preccurlyeq \mu \preccurlyeq \nu$, we have $f_{\lambda\mu} \circ f_{\mu\nu} = f_{\lambda\nu}$

Then $\{M_\lambda, f_{\lambda\mu}\}$ is called an inverse system.

Def. For inverse system $\{M_\lambda, f_{\lambda\mu}\}$, if there exists a module M and a family of R-module maps $\{\varphi_\lambda : M \longrightarrow M_\lambda\}$ such that

1. For any $\lambda \preccurlyeq \mu$, $\varphi_\lambda = f_{\lambda\mu} \varphi_\mu$;
2. $(M, \{\varphi_\lambda\})$ satisfies the following universal property:

If there is an R-module N and R-module maps $\{\psi_\lambda : N \to M_\lambda\}$ s.t. $\forall \lambda \preccurlyeq \mu$, $\psi_\lambda = f_{\lambda\mu} \psi_\mu$, then there is a unique R-module map $\theta : N \to M$ s.t.

$$\psi_\lambda = \varphi_\lambda \theta.$$

$$\begin{tikzcd} & M_\lambda \\ M \arrow[ur, "\varphi_\lambda"] \arrow[dr, "\varphi_\mu"'] & \\ & M_\mu \arrow[uu, "f_{\lambda\mu}"'] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dr, "\varphi_\lambda"] & \\ & M_\lambda \\ N \arrow[uu, dashed, "\theta"] \arrow[ur, "\psi_\lambda"'] & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} M \arrow[rr, dashed, "\exists!\ \theta"] \arrow[dr, "\varphi_\lambda"] \arrow[ddr, "\varphi_\mu"'] & & N \arrow[dl, "\psi_\lambda"'] \arrow[ddl, "\psi_\mu"] \\ & M_\lambda & \\ & M_\mu \arrow[u] & \end{tikzcd}$$

Then M is called inverse limit of $\{M_\lambda\}$, and we denote

$$\varprojlim M_\lambda = M.$$

Prop. In Mod_R, the inverse limit always exists.

proof. Take direct product $\prod_{\alpha \in I} M_\alpha$ and construct submodule as

$$S = \{ x \in \prod_{\alpha \in I} M_\alpha \mid f_{\lambda\mu} \pi_\mu(x) = \pi_\lambda(x),\ \lambda \leq \mu \},$$

then $\varprojlim M_\lambda = \prod_{\alpha \in I} M_\alpha / S$.

Complete this by yourself.

Example. Let $I = \mathbb{Z}_{\geq 0}$, for $i \in I$, define Abelian group $M_i = \mathbb{Z}[x]/(x^i)$.

When $i < j$, there is a canonical homomorphism

$$f_{ij} : M_j \longrightarrow M_i.$$
$$p(x) + (x^j) \longmapsto p(x) + (x^i)$$

This is well defined since $(x^j) \subseteq (x^i)$.

The inverse limit is given by

$$\varprojlim M_i = \mathbb{Z}[[x]]$$

where $\mathbb{Z}[[x]] = \{ a_0 + a_1 x + \cdots + a_n x^n + \cdots \mid a_i \in \mathbb{Z} \}$. The $\varphi_i : \mathbb{Z}[[x]] \to M_i$ is defined as truncation and then taking quotient.

$$\varphi_i : \sum_{k \geq 0} a_k x^k \longmapsto \sum_{k=0}^{i-1} a_k x^k + (x^i)$$

(III) Colimit and limit

- Colimit = direct limit = inductive limit
- limit = inverse limit = projective limit

We will only provide definition of colimit, since limit is a dual concept.

Def (Colimit) Consider two categories $\mathcal{B}$ and $\mathcal{C}$ and a functor

$$F : \mathcal{B} \to \mathcal{C}.$$

We define a new category $\mathcal{D}$:

- $\mathrm{Ob}\,\mathcal{D} = \{ (C', \psi) \}$ where ψ is a family of morphisms

$$\psi = \{ \psi_B : F(B) \longrightarrow C' \mid \psi_B = \psi_{B'} \circ F(f_{B'B}),\ \forall f_{B'B} \in \mathrm{Hom}(B, B') \}$$

$$\begin{tikzcd}
F(B) \arrow[dd, "F(f_{B'B})"'] \arrow[dr, "\psi_B"] & \\
& C \\
F(B') \arrow[ur, "\psi_{B'}"'] &
\end{tikzcd}$$

- Morphism between (C, ψ) and (C', ψ') is defined as

$$g \in \mathrm{Hom}(C, C')$$

such that $\psi'_B = g\psi_B$ for all $B \in \mathrm{Ob}\,\mathcal{B}$.

$$\begin{tikzcd}
F(B) \arrow[dd, "F(f_{B'B})"'] \arrow[dr, "\psi_B"] \arrow[drr, "\psi'_B"] & & \\
& C \arrow[r, "g"] & C' \\
F(B') \arrow[ur, "\psi_{B'}"] \arrow[urr, "\psi'_{B'}"'] & &
\end{tikzcd}$$

If $\mathcal{D}$ has an initial object A, then A will be called colimit

$$\varinjlim F = A.$$

Since initial object, if exists, must be unique; colimit, if exists, must be unique.

Theorem 6.1. If $\mathcal{C}$ is a category for which coproduct exists for any objects, any $A \underset{g}{\overset{f}{\rightrightarrows}} B$ has coequalizer, then, when $\mathcal{B}$ is small category, the colimit of functor $F : \mathcal{B} \to \mathcal{C}$ exists!

For a functor $G : \mathcal{C} \to C$, if it preserves coproduct and coequalizer, then it preserves colimit.