Problem 1 Consider the following classical trajectory of an open string

$$\begin{aligned} X^0 &= B\tau, \\ X^1 &= B\cos\tau\cos\sigma, \\ X^2 &= B\sin\tau\cos\sigma, \\ X^i &= 0, \ \ i > 2, \end{aligned}$$

and assume the conformal gauge condition.

(i) Show that this configuration describes a solution to the equations of motion for the field X^μ corresponding to an open string with Neumann boundary conditions. Show that the ends of this string are moving with the speed of light.

(ii) Compute the energy $E = P^0$ and angular momentum J of the string. Use your result to show that

$$\frac{E^2}{|J|} = 2\pi T = \frac{1}{\alpha'}.$$

(iii) Show that the constraint equation $T_{\alpha\beta} = 0$ can be written as

$$(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0, \ \ \partial_\tau X^\mu \partial_\sigma X_\mu = 0,$$

and that this constraint is satisfied by the above solution.

Solution.

(i) In the conformal gauge condition, the metric is $h_{\alpha\beta} = e^{\phi(\sigma,\tau)}\eta_{\alpha\beta}$ and from Weyl symmetry, take $h_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-,+)$, the equation of motion is

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu = 0. \tag{1}$$

For $\mu = 0$, we have $\partial_\sigma^2 X^0 = 0$ and $\partial_\tau^2 X^0 = 0$, thus equation of motion (1) is satisfied. For $\mu = 1$, we have $\partial_\sigma^2 X^1 = -B\cos\tau\cos\sigma$ and $\partial_\tau^2 X^1 = -B\cos\tau\cos\sigma$, this implies that $(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^1 = 0$. For $\mu = 2$, we have $\partial_\sigma^2 X^2 = -B\sin\tau\cos\sigma$ and $\partial_\tau^2 X^2 = -B\sin\tau\cos\sigma$, this implies that $(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^2 = 0$. For $\mu > 2$, $X^\mu = 0$ the equation of motion is trivially satisfied.

The Neumann boundary condition is $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$, this is trivial for the case X^0 and $X^\mu, \mu > 2$. For $\mu = 1$, $(\partial_\sigma X^1)|_{\sigma=0,\pi} = (-B\cos\tau\sin\sigma)_{\sigma=0,\pi} = 0$. For $\mu = 2$, $(\partial_\sigma X^2)|_{\sigma=0,\pi} = (-B\sin\tau\sin\sigma)_{\sigma=0,\pi} = 0$.

Recall that the speed of a string at point σ is

$$v(\sigma) = c\sqrt{(\frac{dX^1}{dX^0})^2 + (\frac{dX^2}{dX^0})^2}, \tag{2}$$

where c is the speed of light.

$$\begin{aligned} \frac{dX^1}{dX^0} &= -\cos\sigma\sin\tau, \\ \frac{dX^2}{dX^0} &= \cos\sigma\sin\tau, \end{aligned}$$

this implied that

$$v(\sigma) = c\sqrt{cos^2\sigma\sin^2\tau + \cos^2\sigma\cos^2\tau} = c|\cos\sigma|, \tag{3}$$

Thus for the endpoints of string $\sigma = 0, \pi$, $v = c$.

(ii)

$$E = P^0 = T\int_0^\pi d\sigma \partial_\tau X^0 = TB\pi$$

$$\begin{aligned} J^{12} &= T\int_0^{\pi} d\sigma(X^1\partial_\tau X^2 - X^2\partial_\tau X^1 \\ &= T\int_0^{\pi} d\sigma(B^2\cos^\tau\cos^2\sigma + B^2\sin^\tau\cos^2\sigma) \\ &= TB^2\int_0^{\pi} d\sigma\cos^2\sigma \\ &= \frac{\pi}{2}TB^2 \end{aligned} \tag{4}$$

This implies that

$$\frac{E^2}{|J|} = \frac{T^2B^2\pi^2}{\frac{\pi}{2}TB^2} = 2\pi T = \frac{1}{\alpha'}.$$

(iii) Note that the stress-energy tensor T_α is the variation of Lagrangian with respect the metric $g^{\alpha\beta}$ with a particular choice of normalization factor.

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta\mathcal{L}}{\delta g^{\alpha\beta}},$$

where $\mathcal{L} = -\frac{T}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X$. Using the formula for variation of the determinant of a matrix M,

$$\delta\det M = \det M\mathrm{Tr}(M^{-1}\delta M),$$

we have

$$T_{\alpha\beta} = \partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X \tag{5}$$

Note that, here $h_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-,+)$. The equation of motion associated to the metric $h^{\alpha\beta}$ is simply $T_{\alpha\beta} = 0$.

$$\begin{aligned} T_{00} &= \partial_\tau X\cdot\partial_\tau X - \eta_{00}\eta^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X \\ &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} T_{11} &= \partial_\sigma X\cdot\partial_\sigma X - \eta_{11}\eta^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X \\ &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} T_{01} &= \partial_\tau X\cdot\partial_\sigma X - \eta_{01}\eta^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X \\ &= \dot{X}\cdot X' = 0 \end{aligned} \tag{8}$$

We thus completes the proof. To check that the trajectory given above is a solution, consider

$$\dot{X}^\mu = (B, -B\sin\tau\cos\sigma, B\cos\tau\cos\sigma, 0, \cdots, 0) \tag{9}$$

$$X^{\mu\prime} = (0, -B\cos\tau\sin\sigma, -B\sin\tau\sin\sigma, 0, \cdots, 0) \tag{10}$$

We see that $\dot{X}\cdot X' = 0$ and $\dot{X}^2 + X'^2 = 0$ □

Problem 2 Consider the following classical trajectory of an open string

$$\begin{aligned} X^0 &= 3A\tau \\ X^1 &= A\cos 3\tau\cos 3\sigma \\ X^2 &= A\sin a\tau\cos b\sigma \end{aligned}$$

and assume the conformal gauge.

Problem 2 continued on next page. . .

(i) Determine the values of a and b so that the above equations describe an open string that solves the constraint $T_{\alpha\beta} = 0$. Express the solution in the form

$$X^\mu = X_L^\mu(\sigma^-) + X_R^\mu(\sigma^+).$$

Determine the boundary conditions satisfied by this field configuration.

(ii) Plot the solution in (X^1, X^2)-plane as a function of τ in steps of $\pi/12$.

(iii) Compute the center-of-mass momentum and angular momentum and show that they are conserved. What do you obtain for the relation between the energy and angular momentum of this string? Comment on your result.

Solution

(i) As we have shown in problem 1 (iii), the equation $T_{\alpha\beta} = 0$ is equivalent to

$$\dot{X} \cdot X' = 0 \tag{11}$$

$$\dot{X}^2 + X'^2 = 0 \tag{12}$$

Since $\dot{X} = (3A, -3A\sin 3\tau \cos 3\sigma, aA\cos a\tau \cos b\sigma)$ and $X' = (0, -3A\cos 3\tau \sin 3\sigma, -bA\sin a\tau \sin b\sigma)$, substituting them into Eqs. (11) and (12), we have

$$9A^2 \sin 3\tau \cos 3\tau \sin 3\sigma \cos 3\sigma - abA^2 \cos a\tau \sin a\tau \cos b\sigma \sin b\sigma = 0 \tag{13}$$

$$\begin{aligned} &-9A^2 + 9A^2(\sin 3\tau \cos 3\sigma)^2 + a^2A^2(\cos a\tau \cos b\sigma)^2 \\ &+ 9A^2(\cos 3\tau \sin 3\sigma)^2 + b^2A^2(\sin a\tau \sin b\sigma)^2 = 0 \end{aligned} \tag{14}$$

We see that $a = b = 3$ is a solution of the above equations.

For the boundary condition, we see that $X' = (0, -3A\cos 3\tau \sin 3\sigma, -bA\sin a\tau \sin b\sigma)$ which takes zero value at $\sigma = 0, \pi$, thus satisfies the Neumann condition.

(ii)Here we plot the solution in (X^1, X^2)-plane for $\tau = 0, \pi/12, \pi/6, \pi/4$, see Fig. 1.

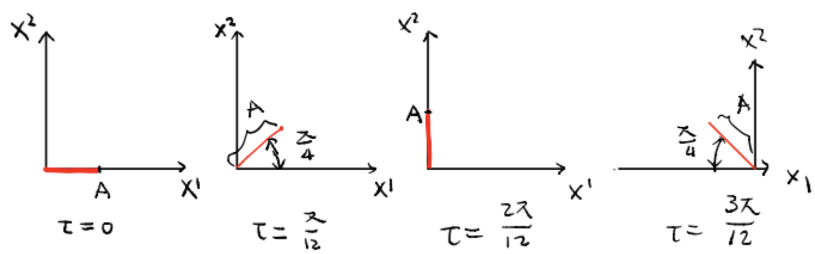

Figure 1: The plot of solution in (X^1, X^2)-plane for $\tau = 0, \pi/12, \pi/6, \pi/4$

(iii) The momenta of string are

$$\mathcal{P}_\mu^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T\frac{\left(\dot{X} \cdot X'\right) X'_\mu - \left(X'^2\right) \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}, \tag{15}$$

$$\mathcal{P}_\mu^\sigma = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T\frac{\left(\dot{X} \cdot X'\right) \dot{X}_\mu - \left(\dot{X}^2\right) X'_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}. \tag{16}$$

and the equation of motion is $\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0$. The center-of-mass momentum is the integration of $\mathcal{P}_\mu^\tau$ over string

$$p^\mu(\tau) = \int_0^\pi \mathcal{P}_\mu^\tau d\sigma \tag{17}$$

To prove it is conserved, consider

$$\frac{d}{d\tau} p^\mu(\tau) = \int_0^\pi \partial_\tau \mathcal{P}_\mu^\tau d\sigma = \int_0^\pi -\partial_\sigma \mathcal{P}_\mu^\sigma d\sigma = -\mathcal{P}_\mu^\sigma|_0^\pi = 0. \tag{18}$$

Note that here we have used the equation of motion and the boundary condition with which we have $\mathcal{P}_\mu^\sigma = 0$ at endpoints of string. The current corresponds to the Lorentz invariance is

$$\mathcal{M}_{\mu\nu}^\alpha = X_\mu \mathcal{P}_\nu^\alpha - X_\nu \mathcal{P}_\mu^\alpha \tag{19}$$

the corresponding equation of motion is

$$\frac{\partial \mathcal{M}_{\mu\nu}^\tau}{\partial \tau} + \frac{\partial \mathcal{M}_{\mu\nu}^\sigma}{\partial \sigma} = 0 \tag{20}$$

From the current we can define the Lorentz charge

$$M_{\mu\nu} = \int \mathcal{M}_{\mu\nu}^\tau(\tau,\sigma) d\sigma = \int \left(X_\mu \mathcal{P}_\nu^\tau - X_\nu \mathcal{P}_\mu^\tau \right) d\sigma \tag{21}$$

The conservation law can be proved in a similar way as momentum.

$$\frac{d}{d\tau} M_{\mu\nu} = \int \frac{\partial}{\partial \tau} \mathcal{M}_{\mu\nu}^\tau(\tau,\sigma) d\sigma = -\int \frac{\partial}{\partial \sigma} \mathcal{M}_{\mu\nu}^\sigma(\tau,\sigma) d\sigma = \mathcal{M}_{\mu\nu}^\sigma(\tau,\sigma)|_{\sigma=0}^{\sigma=\pi} = 0 \tag{22}$$

where we have used the fact that $\mathcal{M}_{\mu\nu}^\sigma$ is zero at string endpoints. The angular momentum is M_{ij}. For the given solution of sting, the only non-vanishing component of the angular momentum is M^{12}. By the similar calculation as in Problem 1, we see that $J = |M^{12}|$ and energy satisfy

$$J = \alpha' E^2. \tag{23}$$

□

Problem 3 Derive the equation of motion of Nambu-Goto action and show that

$$\begin{aligned} X^0 &= 2A\tau \\ X^1 &= A\cos 2\tau \cos 2\sigma \\ X^2 &= A\sin 2\tau \cos 2\sigma \\ X^i &= 0 (i > 2) \end{aligned} \tag{24}$$

is a solution of the equation of motion.

Solution

The Nambu-Goto action is $S = -T \int d\sigma^2 \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}$, the Lagrangian is

$$\mathcal{L} = -T\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2},$$

from which we can calculate the momenta

$$\mathcal{P}_\mu^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -T \frac{\left(\dot{X} \cdot X'\right) X'_\mu - \left(X'^2\right) \dot{X}_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}, \tag{25}$$

$$\mathcal{P}_\mu^\sigma = \frac{\partial \mathcal{L}}{\partial X'^\mu} = -T \frac{\left(\dot{X} \cdot X'\right) \dot{X}_\mu - \left(\dot{X}^2\right) X'_\mu}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \dot{X}^2 X'^2}}. \tag{26}$$

Problem 3 continued on next page. . .

Then using the Euler-Lagrange equation $\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} = \frac{\partial \mathcal{L}}{\partial X^\mu}$, we obtain the equation of motion

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0. \tag{27}$$

Or equivalently $\mathcal{L} = -T\sqrt{-\det G_{\alpha\beta}} = -T\sqrt{-G}$ take the variation $\delta\mathcal{L}/\delta\partial_\alpha X^\mu$. Recall that the the variation of a determinant $\det M$ of matrix M is

$$\delta \det M = \det M \ \mathrm{tr}(M^{-1}\delta M) = -\det M \ \mathrm{tr}(M \delta M^{-1}).$$

We have

$$\partial_\alpha \frac{\delta \mathcal{L}}{\delta \partial_\alpha X^\mu} = 0, \tag{28}$$

where

$$\begin{aligned} \delta\mathcal{L} &= -T\delta\sqrt{-G} \\ &= -T\frac{1}{2}\sqrt{-G}G^{\alpha\beta}\delta G_{\alpha\beta} \\ &= (-T\sqrt{-G}G^{\alpha\beta}\partial_\beta X_\mu)\delta\partial_\alpha X^\mu. \end{aligned} \tag{29}$$

Thus the equation of motion is

$$\partial_\alpha(-T\sqrt{-G}G^{\alpha\beta}\partial_\beta X_\mu) = 0. \tag{30}$$

Dropping the constant factor $-T$, we have

$$\partial_\alpha(\sqrt{-G}G^{\alpha\beta}\partial_\beta X_\mu) = 0, \ \ G = \det G_{\alpha\beta}. \tag{31}$$

To show that

$$\begin{aligned} X^0 &= 2A\tau \\ X^1 &= A\cos 2\tau \cos 2\sigma \\ X^2 &= A \sin 2\tau \cos 2\sigma \\ X^i &= 0 (i > 2) \end{aligned} \tag{32}$$

is a solution. We choose to check the Eq. (31). To this end, let us first calculate

$$\partial_\tau X^\mu = (2A, -2A\sin 2\tau\cos 2\sigma, 2A\cos 2\tau \cos 2\sigma, 0, \cdots, 0) \tag{33}$$
$$\partial_\sigma X^\mu = (0, -2A\cos 2\tau \sin 2\sigma, -2A \sin 2\tau \sin 2\sigma, 0, \cdots, 0) \tag{34}$$

and the metric tensor

$$G_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X = \begin{pmatrix} -4A^2\sin^2 2\sigma & 0 \\ 0 & 4A^2 \sin^2 2\sigma \end{pmatrix} \tag{35}$$

The determinant and inverse matrix of $G_{\alpha\beta}$ is thus obvious since $G_{\alpha\beta}$ is diagonal. Then for each vector

$$\begin{pmatrix} \partial_\tau X^\mu \\ \partial_\sigma X^\mu \end{pmatrix} = \begin{pmatrix} 2A \\ 0 \end{pmatrix}, \begin{pmatrix} -2A\sin 2\tau \cos 2\sigma \\ -2A\cos 2\tau \sin 2\sigma \end{pmatrix}, \begin{pmatrix} 2A\cos 2\tau\cos 2\sigma \\ -2A\sin 2\tau \sin 2\sigma \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{36}$$

we have

$$(\partial_\tau, \partial_\sigma) \begin{pmatrix} \sqrt{-G}(-4A^2\sin^2 2\sigma)^{-1} & 0 \\ 0 & \sqrt{-G}(4A^2\sin^2 2\sigma)^{-1} \end{pmatrix} \begin{pmatrix} \partial_\tau X^\mu \\ \partial_\sigma X^\mu \end{pmatrix} = 0 \tag{37}$$

for $\mu = 0, 1, \cdots, D-1$ as expected. □