Problem 1 Compute the mode expansion of an open string with Neumann boundary conditions for the coordinates $X^0, \dots, X^{24}$, while the remaining coordinate X^{25} satisfies the following boundary conditions:

(i) Dirichlet boundary conditions at both ends

$$X^{25}(0,\tau) = X_0^{25} \quad \text{and} \quad X^{25}(\pi,\tau) = X_\pi^{25}$$

What is the interpretation of such a solution? Compute the conjugate momentum P^{25}. Is this momentum conserved?

(ii) Dirichlet boundary conditions on one end and Neumann boundary conditions at the other end

$$X^{25}(0,\tau) = X_0^{25} \quad \text{and} \quad \partial_\sigma X^{25}(\pi,\tau) = 0$$

What is the interpretation of this solution?

Solution.

Since the equations of motion is $\partial^2 X^\mu = 0$, we see that they are 26 independent equations. Thus they can be solved in their own boundary conditions for each μ. For $\mu = 0, \cdots, 24$, they are open string with Neumann boundary condition, thus the general solution is of the form

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{1}{m} \alpha_m^\mu e^{-im\tau}\cos(m\sigma). \tag{1}$$

Now consider the case for $\mu = 25$.

(i)The Dirichlet boundary condition $X^{25}(0,\tau) = X_0^{25}$ and $X^{25}(\pi,\tau) = X_\pi^{25}$. The equation of motion is $\partial^2 X^{25} = 0$.

The solution is familiar for us from partial differential equation course,

$$X^{25} = F(\sigma^+) + G(\sigma^-).$$

From boundary condition at $\sigma = 0$, $X^{25}(0,\tau) = X_0^{25}$, we see $F(\tau) + G(\tau) = X_0^{25}$, thus $G(\tau) = X_0^{25} - F(\tau)$. Now we can set the solution to be

$$X^{25}(\sigma,\tau) = F(\sigma^+) - F(\sigma^-) + X_0^{25}.$$

Now from the boundary condition at $\sigma = \pi$, $X^{25}(\pi,\tau) = X_\pi^{25}$, we see $F(\tau+\pi) - F(\tau+\pi) - X_0^{25} = X_\pi^{25}$. Here, to introduce the Fourier expansion, we must first construct a periodic function $H(\lambda) := F(\lambda - \pi) + \frac{(X_0^{25} - X_\pi^{25})\lambda}{2\pi}$, or equivalently $F(\lambda) = H(\lambda+\pi) - \frac{(X_0^{25}-X_\pi^{25})(\lambda+\pi)}{2\pi}$. It is easily checked that $H(\lambda + 2\pi) = H(\lambda)$, thus we can expand $H(\lambda)$ as

$$H(\lambda) = \sum_{n\in\mathbb{Z}} \beta_n e^{-in\lambda}.$$

To make $H(\lambda)$ real, we have $\beta_{-n} = \beta_n^*$. Thus $F(\lambda) = H(\lambda+\pi) - \frac{(X_0^{25}-X_\pi^{25})(\lambda+\pi)}{2\pi} = \sum_{n\in\mathbb{Z}} \beta_n e^{-in(\lambda+\pi)} - \frac{(X_0^{25}-X_\pi^{25})(\lambda+\pi)}{2\pi}$.

Now we are at a position to give the general solution.

$$\begin{aligned} X^{25}(\sigma,\tau) =& F(\sigma^+) - F(\sigma^-) - X_0^{25} \\ =& \sum_{n\in\mathbb{Z}} \beta_n e^{-in(\sigma^+ + \pi)} - \frac{(X_0^{25} - X_\pi^{25})(\sigma^+ + \pi)}{2\pi} - \sum_{n\in\mathbb{Z}} \beta_n e^{-in(\sigma^- + \pi)} - \frac{(X_0^{25} - X_\pi^{25})(\sigma^- + \pi)}{2\pi} + X_0^{25} \\ =& -2i \sum_{n\neq 0} \beta_m (-1)^n e^{-in\tau} \sin(n\sigma) + X_0^{25} + \frac{X_\pi^{25} - X_0^{25}}{\pi}\sigma \end{aligned} \tag{2}$$

By setting $\alpha_n := -2i\beta_n(-1)^n n/\sqrt{2\alpha'}$, we get the general solution

$$X^a = X_0^{25} + \frac{X_\pi^{25} - X_0^{25}}{\pi}\sigma + \sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n e^{-in\tau} \sin(n\sigma). \tag{3}$$

Problem 1 continued on next page...

(ii) To make things more convenient, we introduce $\tilde{\sigma} = \pi - \sigma$. Then the form of equation of motion remains unchanged $(\partial_{\tilde{\sigma}}^2 - \partial_\tau^2)X^{25}(\tilde{\sigma},\tau) = 0$. Firstly, $X^{25}(\tilde{\sigma},\tau) = F(\tilde{\sigma}^+) + G(\tilde{\sigma}^-)$ and $X^{25\prime} = F'(\tilde{\sigma}^+) - G'(\tilde{\sigma}^-)$. The right boundary condition becomes $\partial_{\tilde{\sigma}} X^{25}(\tilde{\sigma} = 0, \tau) = 0$, we see that $F'(\tau) - G'(\tau) = 0$, hereinafter the derivative is for $\tilde{\sigma}$. This boundary condition implies that we can take $G(\lambda) = F(\lambda) + C$. Thus the solution can be taken as

$$X^{25}(\tilde{\sigma},\tau) = F(\tilde{\sigma}^+) + F(\tilde{\sigma}^-) + C$$

The left boundary condition is now $X^{25}(\tilde{\sigma} = \pi, \tau) = X_0^{25}$. This implies that $F(\tau+\pi) + F(\tau - \pi) + C = X_0^{25}$. In a similar way as in (i), we introduce a new function $H(\lambda) := F(\lambda - \pi) + \frac{C - X_0^{25}}{2}$. It's easily checked that $H(\lambda + 2\pi) + H(\lambda) = 0$, thus $H(\lambda + 4\pi) = -H(\lambda + 2\pi) = H(\lambda)$, thus it is a periodic function with period 4π. Thus we can take the Fourier expansion as

$$H(\lambda) = \sum_{n\in\mathbb{Z}} \beta_n e^{-in\lambda/2}.$$

Then from $H(\lambda + 2\pi) + H(\lambda) = 0$,

$$H(\lambda) = \sum_{n\in\mathbb{Z}} \beta_n e^{-in\pi} e^{-in\lambda/2} = -\sum_{n\in\mathbb{Z}} \beta_n e^{-in\lambda/2},$$

this implies that $\beta_n = 0$ for n even. To make $H(\lambda)$ real, we also have $\beta_{-n} = \beta_n^*$. Therefore $H(\lambda) = \sum_{n\in\mathbb{Z}_{\text{odd}}} \beta_n e^{-in\lambda/2}$.

We now at a position to give the general solution. Since $F(\lambda) = H(\lambda) - \frac{C - X_0^{25}}{2} = \sum_{n\in\mathbb{Z}_{\text{odd}}} \beta_n e^{-in(\lambda+\pi)/2} - \frac{C - X_0^{25}}{2}$, we have

$$\begin{aligned} X(\tilde{\sigma},\tau) &= F(\tilde{\sigma}^+) + F(\tilde{\sigma}^-) + C \\ &= X_0^{25} + \sum_{n\in\mathbb{Z}_{\text{odd}}} \beta_n e^{-in(\tilde{\sigma}^+ + \pi)/2} + \sum_{n\in\mathbb{Z}_{\text{odd}}} \beta_n e^{-in(\tilde{\sigma}^- + \pi)/2} \\ &= X_0^{25} + \sum_{n\in\mathbb{Z}_{\text{odd}}} e^{-in\pi/2} \beta_n e^{-in\tau/2} 2\cos(n\tilde{\sigma}/2) \end{aligned} \tag{4}$$

Now we can define $\alpha_n = -2in\frac{e^{-in\pi/2}}{\sqrt{2\alpha'}}\beta_n$, we obtain the general solution

$$\begin{aligned} X^{25}(\sigma,\tau) &= X_0^{25} + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}_{\text{odd}}} \frac{1}{n} \alpha_n e^{-in\tau/2} \cos(n\tilde{\sigma}/2) \\ &= X_0^{25} + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}_{\text{odd}}} \frac{1}{n} \alpha_n e^{-in\tau/2} \cos(n(\pi - \sigma)/2) \\ &+ X_0^{25} + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}_{\text{odd}}} \frac{1}{n} \alpha_n e^{-in\tau/2} e^{i(n-1)\pi/2} \sin(n\sigma/2) \end{aligned} \tag{5}$$

□

Problem 2 Use the mode expansion of an open string with Neumann boundary conditions in Eq. (2.62) and the commutation relations for the modes in Eq. (2.54) to check explicitly the equal-time commutators

$$\left[X^\mu(\sigma,\tau), X^\nu\left(\sigma',\tau\right)\right] = \left[P^\mu(\sigma,\tau), P^\nu\left(\sigma',\tau\right)\right] = 0$$

while

$$\left[X^\mu(\sigma,\tau), P^\nu\left(\sigma',\tau\right)\right] = i\eta^{\mu\nu}\delta\left(\sigma - \sigma'\right)$$

Hint: The representation $\delta\left(\sigma-\sigma^{\prime}\right)=\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \cos (n \sigma) \cos \left(n \sigma^{\prime}\right)$ might be useful.

Solution.

Recall the commutators

$$
\begin{aligned}
& [\alpha_m^\mu, \alpha_n^\nu]=m \eta^{\mu \nu} \delta_{m+n, 0}, && (6)\\
& [x^\mu, p^\nu]=i \eta^{\mu \nu} && (7)
\end{aligned}
$$

with all others zero. The momentum $P^\mu=T \dot{X}^\mu$. The mode expansion of an open string with Neumann boundary conditions is

$$
X^\mu(\tau, \sigma)=x^\mu+l_{\mathrm{s}}^2 p^\mu \tau+i l_{\mathrm{s}} \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-i m \tau} \cos (m \sigma), \tag{8}
$$

the momentum takes the form

$$
P^\mu(\tau, \sigma)=T l_{\mathrm{s}} \sum_{m \in \mathbb{Z}} \alpha_m^\mu e^{-i m \tau} \cos (m \sigma), \tag{9}
$$

where $\alpha_0^\mu=l_{\mathrm{s}} p^\mu$, $T=1 /(2 \pi \alpha^{\prime})$ and $l_s^2=2 \alpha^{\prime}$.

Firstly, for $\left[X^\mu(\sigma, \tau), X^\nu\left(\sigma^{\prime}, \tau\right)\right]$, in the mode expansion form

$$
\begin{aligned}
& [x^\mu+l_{\mathrm{s}}^2 p^\mu \tau+i l_{\mathrm{s}} \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu e^{-i m \tau} \cos (m \sigma), x^\nu+l_{\mathrm{s}}^2 p^\nu \tau+i l_{\mathrm{s}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\nu e^{-i n \tau} \cos (n \sigma^{\prime})] \\
= & [x^\mu, l_{\mathrm{s}}^2 p^\nu]+[l_{\mathrm{s}}^2 p^\mu, x^\nu]+l_{\mathrm{s}} \sum_{n \neq 0} \frac{1}{n} e^{-i n \tau} \cos (n \sigma^{\prime})[\alpha_0^\mu, \alpha_n^\nu]+l_{\mathrm{s}} \sum_{m \neq 0} \frac{1}{m} e^{-i m \tau} \cos (m \sigma^{\prime})[\alpha_m^\mu, \alpha_0^\nu] \\
& +\sum_{m \neq 0} \sum_{n \neq 0} \frac{1}{m n} e^{-i(m+n) \tau} \cos m \sigma \cos n \sigma^{\prime}[\alpha_m^\mu, \alpha_n^\nu] \\
= & \sum_{m \neq 0} \sum_{n \neq 0} \frac{1}{m n} e^{-i(m+n) \tau} \cos m \sigma \cos n \sigma^{\prime}[\alpha_m^\mu, \alpha_n^\nu] \\
= & \sum_{m \neq 0} \sum_{n \neq 0} \frac{1}{m n} e^{-i(m+n) \tau} \cos m \sigma \cos n \sigma^{\prime} m \eta^{\mu \nu} \delta_{m+n, 0} \\
= & \eta^{\mu \nu} \sum_{n \neq 0} \frac{1}{n} \cos (-n \sigma)(\cos n \sigma^{\prime}) \\
= & \eta^{\mu \nu}(\sum_{n=1}^{+\infty} \frac{1}{n} \cos (-n \sigma) \cos n \sigma^{\prime}-\sum_{n=1}^{+\infty} \frac{1}{n} \cos (-n \sigma) \cos n \sigma^{\prime}) \\
= & 0,
\end{aligned} \tag{10}
$$

where we have used the property that $\frac{1}{n} \cos (-n \sigma)(\cos n \sigma^{\prime})$ is an odd function with respect to n.

Secondly, for $\left[P^\mu(\sigma, \tau), P^\nu\left(\sigma^{\prime}, \tau\right)\right]$, in the mode expansion form

$$
\begin{aligned}
& [T l_{\mathrm{s}} \sum_{m \in \mathbb{Z}} \alpha_m^\mu e^{-i m \tau} \cos (m \sigma), T l_{\mathrm{s}} \sum_{n \in \mathbb{Z}} \alpha_n^\nu e^{-i n \tau} \cos (n \sigma^{\prime})] \\
= & T^2 l_{\mathrm{s}}^2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} e^{-i m \tau} \cos (m \sigma) e^{-i n \tau} \cos (n \sigma^{\prime})[\alpha_m^\mu, \alpha_n^\nu] \\
= & T^2 l_{\mathrm{s}}^2 \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} e^{-i m \tau} \cos (m \sigma) e^{-i n \tau} \cos (n \sigma^{\prime}) m \eta^{\mu \nu} \delta_{m+n, 0} \\
= & T^2 l_{\mathrm{s}}^2 \eta^{\mu \nu} \sum_{n \in \mathbb{Z}}(-n) \cos (-n \sigma) \cos n \sigma^{\prime} \\
= & 0,
\end{aligned} \tag{11}
$$

Problem 2 continued on next page. . .

here we again used the property that $(-n)\cos(-n\sigma)\cos n\sigma'$ is an odd function with respect to n.

Finally, for the case of $\left[X^\mu(\sigma,\tau), P^\nu\left(\sigma',\tau\right)\right]$, expanded in the oscillation modes

$$\begin{aligned}
&[x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0}\frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos(m\sigma), T l_s \sum_{n\in\mathbb{Z}}\alpha_n^\nu e^{-in\tau}\cos(n\sigma')] \\
=&[x^\mu, T l_s \alpha_0^\nu] + l_s^2 T\tau \sum_{n\in\mathbb{Z}} e^{-in\tau}\cos(n\sigma')[\alpha_0^\mu, \alpha_n^\nu] \\
&+ i l_s^2 T \sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{m}e^{-im\tau}\cos(m\sigma)e^{-in\tau}\cos(n\sigma')[\alpha_m^\mu,\alpha_n^\nu] \\
=&T l_s^2[x^\mu, p^\nu] + i l_s^2 T \sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{m}e^{-im\tau}\cos(m\sigma)e^{-in\tau}\cos(n\sigma')m\eta^{\mu\nu}\delta_{m+n,0} \\
=&T l_s^2 i\eta^{\mu\nu} + i l_s^2 T\eta^{\mu\nu}\sum_{m\neq 0}\cos(m\sigma)\cos(-m\sigma') \\
=&T l_s^2 i\eta^{\mu\nu}\sum_{m\in\mathbb{Z}}\cos(m\sigma)\cos(-m\sigma') \\
=&T l_s^2 i\eta^{\mu\nu}\pi\delta(\sigma-\sigma') \\
=&i\eta^{\mu\nu}\delta(\sigma-\sigma'),
\end{aligned} \tag{12}$$

as we expected. □

Problem 3 Drive the Hamiltonian for closed and open string in mode expansions.

$$H = \sum_{n=-\infty}^{+\infty}\left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right), \tag{13}$$

$$H = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\alpha_{-n}\cdot\alpha_n. \tag{14}$$

Solution.

Firstly, consider $\mathcal{L} = \frac{T}{2}(\dot{X}^2 - X'^2)$, $\mathcal{P}^\mu = \delta\mathcal{L}/\delta X_\mu = T\dot{X}^\mu$, the Hamiltonian density is

$$\mathcal{H} = \dot{X}_\mu\mathcal{P}^\mu - \mathcal{L} = \frac{T}{2}(\dot{X}^2 + X'^2). \tag{15}$$

Let us first consider the closed string case.

$$\begin{aligned}
\dot{X}^\mu &= \partial_+ X_L^\mu(\sigma^+) + \partial_- X_R^\mu(\sigma^-) \\
X^{\mu'} &= \partial_+ X_L^\mu(\sigma^+) - \partial_- X_R^\mu(\sigma^-)
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\partial_- X_R^\mu(\sigma^-) &= l_s \textstyle\sum_{m\in\mathbb{Z}} \alpha_n^\mu e^{-2in\sigma^-} \\
\partial_+ X_L^\mu(\sigma^+) &= l_s \textstyle\sum_{m\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-2in\sigma^+}
\end{aligned} \tag{17}$$

From which we have

$$\dot{X}^2 = l_s^2\sum_{n,m\in\mathbb{Z}}(\tilde{\alpha}_n\cdot\tilde{\alpha}_m e^{-2i(n+m)\sigma^+} + \tilde{\alpha}_n\cdot\alpha_m e^{-2i(n\sigma^+ + m\sigma^-)} + \alpha_n\cdot\tilde{\alpha}_m e^{-2i(n\sigma^- + m\sigma^+)} + \alpha_n\cdot\alpha_m e^{-2i(n+m)\sigma^-})$$

$$X'^2 = l_s^2\sum_{n,m\in\mathbb{Z}}(\tilde{\alpha}_n\cdot\tilde{\alpha}_m e^{-2i(n+m)\sigma^+} - \tilde{\alpha}_n\cdot\alpha_m e^{-2i(n\sigma^+ + m\sigma^-)} - \alpha_n\cdot\tilde{\alpha}_m e^{-2i(n\sigma^- + m\sigma^+)} + \alpha_n\cdot\alpha_m e^{-2i(n+m)\sigma^-})$$

Problem 3 continued on next page...

Thus

$$\mathcal{H} = Tl_s^2 \sum_{n,m\in\mathbb{Z}} (\tilde{\alpha}_n \cdot \tilde{\alpha}_m e^{-2i(n+m)\sigma^+} + \alpha_n \cdot \alpha_m e^{-2i(n+m)\sigma^-})$$

Then integrating over σ and using the formula $\int_0^\pi e^{-2ik\sigma} d\sigma = \pi\delta_{k,0}$, we have

$$\begin{aligned} H &= \int_0^\pi \mathcal{H} d\sigma \\ &= Tl_s^2 \sum_{n,m\in\mathbb{Z}} (\tilde{\alpha}_n \cdot \tilde{\alpha}_m \int_0^\pi e^{-2i(n+m)\sigma^+} d\sigma + \alpha_n \cdot \alpha_m \int_0^\pi e^{-2i(n+m)\sigma^-} d\sigma) \\ &= Tl_s^2 \sum_{n,m\in\mathbb{Z}} (\tilde{\alpha}_n \cdot \tilde{\alpha}_m e^{-2i(n+m)\tau} \pi\delta_{n+m,0} + \alpha_n \cdot \alpha_m e^{-2i(n+m)\tau} \pi\delta_{n+m,0}) \\ &= \sum_{n\in\mathbb{Z}} (\alpha_{-n} \cdot \alpha_n + \widetilde{\alpha}_{-n} \cdot \widetilde{\alpha}_n) \end{aligned} \tag{18}$$

as expected.

For the open string, the procedure is completely the same.

$$2\partial_\pm X^\mu = \dot{X}^\mu \pm X^\mu = l_s \sum_{m=-\infty}^{\infty} \alpha_m^\mu e^{-im(\tau\pm\sigma)}, \tag{19}$$

from which we have

$$\begin{aligned} \dot{X}^\mu &= l_s \sum_{m\in\mathbb{Z}} \alpha_m^\mu e^{-im\tau} \cos m\sigma \\ X^{\mu\prime} &= -il_s \sum_{m\in\mathbb{Z}} \alpha_m^\mu e^{-im\tau} \sin m\sigma \end{aligned} \tag{20}$$

These implies that

$$\begin{aligned} \dot{X}^2 &= l_s^2 \sum_{m,n\in\mathbb{Z}} \alpha_n \cdot \alpha_m e^{-i(m+n)\tau} \cos m\sigma \cos n\sigma \\ X'^2 &= -l_s^2 \sum_{m,n\in\mathbb{Z}} \alpha_n \cdot \alpha_m e^{-i(m+n)\tau} \sin m\sigma \sin n\sigma \end{aligned} \tag{21}$$

Therefore, the Hamiltonian density is

$$\mathcal{H} = \frac{T}{2} l_s^2 \sum_{m,n\in\mathbb{Z}} \alpha_n \cdot \alpha_m e^{-i(n+m)\tau} \cos(m+n)\sigma, \tag{22}$$

Then by integrating over σ and using the formula $\int_0^\pi \cos k\sigma = \pi\delta_{k,0}$, we obtain

$$\begin{aligned} H &= \int_0^\pi \mathcal{H} d\sigma \\ &= \frac{T}{2} l_s^2 \sum_{m,n\in\mathbb{Z}} \alpha_n \cdot \alpha_m e^{-i(n+m)\tau} \int_0^\pi \cos(m+n)\sigma \\ &= \frac{T}{2} l_s^2 \sum_{m,n\in\mathbb{Z}} \alpha_n \cdot \alpha_m e^{-i(n+m)\tau} \pi\delta_{m+n,0} \\ &= \frac{1}{2} \sum_{n\in\mathbb{Z}} \alpha_{-n}\alpha_n \end{aligned} \tag{23}$$

as expected. □

Problem 4 Prove the Poisson bracket for Virasoro generators

$$[L_m, L_n]_{\text{PB}} = -i(m-n)L_{m+n}. \tag{24}$$

Solution.
This can be proved by directly calculating, to do this let us first calculate

$$\begin{aligned}
&[L_m, \alpha_s^\nu]_{\text{PB}} \\
=&\frac{1}{2}\sum_{k\in\mathbb{Z}}[\alpha^\mu_{m-k}(\alpha_\mu)_k, \alpha^\nu_s]_{\text{PB}} \\
=&\frac{1}{2}\sum_{k\in\mathbb{Z}}(\alpha^\mu_{m-k}(-i\eta_\mu{}^\nu k\delta_{k+s,0}) - i(m-k)\eta^{\mu\nu}\delta_{m-k+s,0}(\alpha_\mu)_k) \\
=&is\alpha^\nu_{m+s}
\end{aligned} \tag{25}$$

Now consider the commutator

$$\begin{aligned}
&[L_m, L_n]_{\text{PB}} \\
=&\frac{1}{2}\sum_p [L_m, \alpha^\mu_{n-p}(\alpha_\mu)_p]_{\text{PB}} \\
=&\frac{1}{2}\sum_p \alpha^\mu_{n-p}[L_m, (\alpha_\mu)_p]_{\text{PB}} + [L_m, \alpha^\mu_{n-p}]_{\text{PB}}(\alpha_\mu)_p \\
=&\frac{1}{2}\sum_p(\alpha^\mu_{n-p} ip(\alpha_\mu)_{m+p} + i(n-p)\alpha^\mu_{m+n-p})(\alpha_\mu)_p \\
=&\frac{1}{2}\sum_{p'} i(p'-m)\alpha_{n+m-p'}\cdot\alpha_{p'} + \frac{1}{2}\sum_p i(n-p)\alpha_{m+n-p}\cdot\alpha_p \\
=&-i(m-n)L_{m+n},
\end{aligned} \tag{26}$$

Note that in the last step we have introduced $p' = m + p$ and changed the dummy summation subscript p' to p. □

Problem 5 Drive the formula for Lorentz generators

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right) \tag{27}$$

and the commutator

$$[L_m, M^{\mu\nu}] = 0 \tag{28}$$

Solution.
For Lorentz generators, recall that the density is $\mathcal{M}^{\mu\nu} = X^\mu\mathcal{P}^\nu - X^\nu\mathcal{P}^\mu$

$$M^{\mu\nu} = \int_0^\pi \mathcal{M}^{\mu\nu}d\sigma = T\int_0^\pi \left(X^\mu\dot{X}^\nu - X^\nu\dot{X}^\mu\right)d\sigma$$

Now for the mode expansion of open string

$$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu\tau + il_s\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_m e^{-im\tau}\cos(m\sigma) \tag{29}$$

$$\dot{X}^\nu(\tau,\sigma) = l_s^2 p^\nu + l_s\sum_{n\neq 0}\alpha^\nu_n e^{-in\tau}\cos(n\sigma) \tag{30}$$

Problem 5 continued on next page. . .

and $T=1/\left(\pi l_s^2\right)$. A short calculation gives

$$M^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu-i\sum_{m=1}^{\infty}\frac{1}{m}\left(\alpha_{-m}^\mu\alpha_m^\nu-\alpha_{-m}^\nu\alpha_m^\mu\right)$$

Note that here we have used $\int_0^\pi\cos m\sigma\cos n\sigma=\frac{\pi}{2}(\delta_{m,n}+\delta_{m,-n})$.

For the commutator, recall that $[L_m,\alpha_s^\nu]_{\rm PB}=is\alpha_{m+s}^\nu$ as what we have shown in Eq. (25), quantum mechanically $i[\cdot,\cdot]_{\rm PB}\to[\cdot,\cdot]$, it becomes $[L_m,\alpha_s^\nu]=-s\alpha_{m+s}^\nu$ and we can also calculate $[L_m,x^\nu]=-il_s\alpha_m^\nu$ and $[L_m,p^\nu]=0$. Thus

$$\begin{aligned}[L_m,M^{\mu\nu}]=&[L_m,x^\mu p^\nu]-[L_m,x^\nu p^\mu]-i\sum_{n=1}^{+\infty}\frac{1}{n}[L_m,\alpha_{-n}^\mu\alpha_n^\nu-\alpha_{-n}^\nu\alpha_n^\nu]\\=&-il_s(\alpha_m^\mu p^\nu-\alpha_m^\nu p^\mu)-i\sum_{n=1}^{+\infty}(\alpha_{m-n}^\mu\alpha_n^\nu-\alpha_{-n}^\mu\alpha_{m+n}^\nu+\alpha_{-n}^\nu\alpha_{m+n}^\mu-\alpha_{m-n}^\nu\alpha_n^\mu)\\=&-i(\alpha_m^\mu\alpha_0^\nu-\alpha_m^\nu\alpha_0^\mu)-i\sum_{n=1}^{+\infty}(\alpha_{m-n}^\mu\alpha_n^\nu-\alpha_{-n}^\mu\alpha_{m+n}^\nu+\alpha_{-n}^\nu\alpha_{m+n}^\mu-\alpha_{m-n}^\nu\alpha_n^\mu)\end{aligned}\tag{31}$$

For $m=0$, it is obviously equal to 0.

For $m\neq0$, the right hand side of Eq. (31) is

$$\begin{aligned}\text{RHS}=&-i(\alpha_m^\mu\alpha_0^\nu-\alpha_m^\nu\alpha_0^\mu)-i\sum_{n=1}^{\infty}(\alpha_{m-n}^\mu\alpha_n^\nu-\alpha_{m-n}^\nu\alpha_n^\mu)-i\sum_{n=1}^{\infty}(-\alpha_{-n}^\mu\alpha_{m+n}^\nu+\alpha_{-n}^\nu\alpha_{m+n}^\mu)\\=&-i(\alpha_m^\mu\alpha_0^\nu-\alpha_m^\nu\alpha_0^\mu)-i\sum_{n=1}^{m-1}(\alpha_{m-n}^\mu\alpha_n^\nu-\alpha_{m-n}^\nu\alpha_n^\mu)-i(\alpha_0^\mu\alpha_m^\nu-\alpha_0^\nu\alpha_m^\mu)+\\&[-i\sum_{n=m+1}^{\infty}(\alpha_{m-n}^\mu\alpha_n^\nu-\alpha_{m-n}^\nu\alpha_n^\mu)-i\sum_{n=1}^{\infty}(-\alpha_{-n}^\mu\alpha_{m+n}^\nu+\alpha_{-n}^\nu\alpha_{m+n}^\mu)]\end{aligned}\tag{32}$$

We see that the first term cancels the third term and the second and the last terms both equal to zero, thus we complete the proof. □

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