Problem 1 Consider an on-shell open-string state of the form

$$|\phi\rangle = \left(A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C\left(\alpha_0\cdot\alpha_{-1}\right)^2\right)|0;k\rangle$$

where A, B and C are constants. Determine the conditions on the coefficients A, B and C so that $|\phi\rangle$ satisfies the physical-state conditions for $a = 1$ and arbitrary D. Compute the norm of $|\phi\rangle$. What conclusions can you draw from the result?

Solution. First recall that the mass-shell condition for open string is

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n - a = N - a. \tag{1}$$

Since $|\phi\rangle$ is a physical state, $L_m|\phi\rangle = 0$ for all $m > 0$ and $(L_0 - 1)|\phi\rangle = 0$.

From $[L_m, \alpha_s^\nu] = -s\alpha_{m+s}^\nu$ which I have proved in last problem sheet, we have

$$[L_0, \alpha_k\cdot\alpha_l] = -(k+l)\alpha_k\cdot\alpha_l. \tag{2}$$

Using the above equality, we have

$$[L_0, \alpha_{-1}\cdot\alpha_{-1}] = 2\alpha_{-1}\cdot\alpha_{-1} \tag{3}$$
$$[L_0, \alpha_0\cdot\alpha_{-2}] = 2\alpha_0\cdot\alpha_{-2} \tag{4}$$
$$[L_0, (\alpha_0\cdot\alpha_{-1})^2] = 2\left(\alpha_0\cdot\alpha_{-1}\right)^2 \tag{5}$$

Thus $[L_0, A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C\left(\alpha_0\cdot\alpha_{-1}\right)^2] = 2\left(A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C\left(\alpha_0\cdot\alpha_{-1}\right)^2\right)$, for the condition that $(L_0 - 1)|\phi\rangle = 0$, we have

$$(\frac{1}{2}\alpha_0^2 + 1)|\phi\rangle = 0 \tag{6}$$

This is indeed the mass formula for state $|\phi\rangle$, since the momentum is an independent extra structure of the string state, thus we have

$$(\frac{1}{2}\alpha_0^2 + 1)|0;k\rangle = 0 \tag{7}$$

equivalently $\alpha_0^2 = -2$ over state $|0;k\rangle$.

Now consider the constraints that $L_m|\phi\rangle = 0$ for all $m > 0$. Using the commutator

$$[L_m, \alpha_k\cdot\alpha_l] = -(l\alpha_k\cdot\alpha_{m+l} + k\alpha_{m+k}\cdot\alpha_l) \tag{8}$$

we have

$$[L_m, \alpha_{-1}\cdot\alpha_{-1}] = \alpha_{-1}\cdot\alpha_{m-1} + \alpha_{m-1}\cdot\alpha_{-1} \tag{9}$$
$$[L_m, \alpha_0\cdot\alpha_{-2}] = 2\alpha_0\cdot\alpha_{m-2} \tag{10}$$
$$[L_m, (\alpha_0\cdot\alpha_{-1})^2] = \left(\alpha_0\cdot\alpha_{-1}\right)\left(\alpha_0\cdot\alpha_{m-1}\right) + \left(\alpha_0\cdot\alpha_{m-1}\right)\left(\alpha_0\cdot\alpha_{-1}\right) \tag{11}$$

For $m > 2$, they become

$$[L_m, \alpha_{-1}\cdot\alpha_{-1}] = 2\alpha_{-1}\cdot\alpha_{m-1} \tag{12}$$
$$[L_m, \alpha_0\cdot\alpha_{-2}] = 2\alpha_0\cdot\alpha_{m-2} \tag{13}$$
$$[L_m, (\alpha_0\cdot\alpha_{-1})^2] = 2\left(\alpha_0\cdot\alpha_{-1}\right)\left(\alpha_0\cdot\alpha_{m-1}\right) \tag{14}$$

note that here we have used the commutators between mode operators. These commutators all get zero values on state $|0;k\rangle$, thus

$$L_m|\phi\rangle = \left(A\alpha_{-1}\cdot\alpha_{-1} + B\alpha_0\cdot\alpha_{-2} + C\left(\alpha_0\cdot\alpha_{-1}\right)^2\right)L_m|0;k\rangle = 0 \tag{15}$$

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The constraints are trivial.

For $m = 1$, the commutators become

$$[L_1, \alpha_{-1} \cdot \alpha_{-1}] = 2\alpha_{-1} \cdot \alpha_0 \tag{16}$$

$$[L_1, \alpha_0 \cdot \alpha_{-2}] = 2\alpha_{-1} \cdot \alpha_0 \tag{17}$$

$$[L_1, (\alpha_0 \cdot \alpha_{-1})^2] = 2\alpha_0^2 (\alpha_{-1} \cdot \alpha_0) \tag{18}$$

From which we have

$$\begin{aligned} L_1|\phi\rangle =& \left(A\alpha_{-1} \cdot \alpha_{-1} + B\alpha_0 \cdot \alpha_{-2} + C(\alpha_0 \cdot \alpha_{-1})^2\right) L_1|0;k\rangle \\ &+ \left(2A\alpha_{-1} \cdot \alpha_0 + 2B\alpha_{-1} \cdot \alpha_0 + 2C\alpha_0^2(\alpha_{-1} \cdot \alpha_0)\right)|0;k\rangle \\ =& 2\left(A + B + C\alpha_0^2\right)\alpha_{-1} \cdot \alpha_0|0;k\rangle = 0 \end{aligned} \tag{19}$$

□

For $m = 2$, the commutators become

$$\begin{aligned} [L_2, \alpha_{-1} \cdot \alpha_{-1}] &= [L_2, 2(L_{-2} - \frac{1}{2}\sum_{n\neq -1} \alpha_{-2-n} \cdot \alpha_n)] \\ &= 8L_0 + D - \sum_{n\neq -1} (2+n)\alpha_{-n} \cdot \alpha_n + (-n)\alpha_{-(2+n)} \cdot \alpha_{2+n} \\ &= D + \alpha_{-1} \cdot \alpha_1 + \alpha_1 \cdot \alpha_{-1} \end{aligned} \tag{20}$$

$$[L_2, \alpha_0 \cdot \alpha_{-2}] = 2\alpha_0^2 \tag{21}$$

$$[L_2, (\alpha_0 \cdot \alpha_{-1})^2] = (\alpha_0 \cdot \alpha_{-1})(\alpha_0 \cdot \alpha_1) + \alpha_0^2 + \alpha_0^\mu \alpha_{-1}^\nu \alpha_0^\rho \alpha_1^\delta \eta_{\mu\delta}\eta_{\rho\nu} \tag{22}$$

Using the above commutators, we obtain

$$\begin{aligned} L_2|\phi\rangle =& \left(A\alpha_{-1} \cdot \alpha_{-1} + B\alpha_0 \cdot \alpha_{-2} + C(\alpha_0 \cdot \alpha_{-1})^2\right) L_2|0;k\rangle \\ &+ A(D + \alpha_{-1} \cdot \alpha_1 + \alpha_1 \cdot \alpha_{-1}) + 2B\alpha_0^2 + C((\alpha_0 \cdot \alpha_{-1})(\alpha_0 \cdot \alpha_1) + \alpha_0^2 + \alpha_0^\mu \alpha_{-1}^\nu \alpha_0^\rho \alpha_1^\delta \eta_{\mu\delta}\eta_{\rho\nu})|0;k\rangle \\ =& DA + 2B\alpha_0^2 + C\alpha_0^2|0;k\rangle = 0 \end{aligned} \tag{23}$$

Thus over the state $|0;k\rangle$ we have

$$\begin{cases} \alpha_0^2 = -2 \\ A + B + C\alpha_0^2 = 0 \\ DA + 2B\alpha_0^2 + C\alpha_0^2 = 0 \end{cases} \tag{24}$$

this implies that $B = A(D-1)/5$ and $C = A(D+4)/10$.

The norm of the state is

$$\langle\phi|\phi\rangle = 2\left(DA^2 - 2B^2 - 4AC + 4C^2\right) = \frac{2A^2}{25}(D-1)(26-D), \tag{25}$$

from which we see that the norm is nagative if $D > 26$ and it is zero when $D = 26$. Thus $D = 26$ is the critical spacetime dimension.

Problem 2 Construct the spectrum of open and closed strings in light-cone gauge for level $N = 3$. How many states are there in each case? Without actually doing it (unless you want to), describe a strategy for assembling these states into irreducible $SO(25)$ multiplets.

Solution.

Problem 2 continued on next page…

For level $N = 3$ open string, the states are the following

$$\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0;k\rangle \tag{26}$$

the number of states is $\frac{24\cdot23\cdot22}{6} + 24\cdot 23 + 24 = 2600$.

$$\alpha^i_{-2}\alpha^j_{-1}|0;k\rangle \tag{27}$$

the number of states is $24^2 = 576$.

$$\alpha^i_{-3}|0;k\rangle \tag{28}$$

the number of states is 24. Thus the total number of states are 3200.

For level $N = \tilde{N} = 3$ closed string, the states are the following, the left movers:

$$\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}\tilde{\alpha}^k_{-1}|0;k\rangle \tag{29}$$

the number of states is $\frac{24\cdot23\cdot22}{6} + 24\cdot 23 + 24 = 2600$.

$$\tilde{\alpha}^i_{-2}\tilde{\alpha}^j_{-1}|0;k\rangle \tag{30}$$

the number of states is $24^2 = 576$.

$$\tilde{\alpha}^i_{-3}|0;k\rangle \tag{31}$$

the number of states is 24.

The right movers

$$\alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0;k\rangle \tag{32}$$

the number of states is $\frac{24\cdot23\cdot22}{6} + 24\cdot 23 + 24 = 2600$.

$$\alpha^i_{-2}\alpha^j_{-1}|0;k\rangle \tag{33}$$

the number of states is $24^2 = 576$.

$$\alpha^i_{-3}|0;k\rangle \tag{34}$$

the number of states is 24.

Thus the total number of level $N = \tilde{N} = 3$ closed string states are 3200×3200.

Let us now discuss the strategy to assemble these states into irreducible $SO(25)$ multiplets. We choose to talk open string states (since the closed string corresponds just to the double of the group, left movers and right movers form independent representations). As we known that $[L_m, M^{\mu\nu}] = 0$ for Lorentz generators $M^{\mu\nu}$, this implies that the physical state condition is invariant under Lorentz transformations. Every physical state is mapped again into physical state by Lorentz transformation, and the mass of states doesn't change. Therefore physical states are grouped into Lorentz multiplets. The level $N = 3$ open string states can be assembled into $\mathbf{2900} \otimes \mathbf{300}$ representation of $SO(25)$. This is because that the number of components of symmetric traceless rank-3 tensor with $SO(25)$ indices is 2900, and the number of components of rank-2 antisymmetric tensor with $SO(25)$ indices is 300. In this way we group the 3200 states level $N = 3$ open string states into two sets.

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