Problem 1 Prove that the generators of Poincaré group obey

$$\begin{aligned}
\left[D, P_\mu\right] &= iP_\mu, \left[D, K_\mu\right] = -iK_\mu, \left[K_\mu, P_\nu\right] = 2i\left(\eta_{\mu\nu}D + J_{\mu\nu}\right)\\
\left[K_\mu, J_{\nu\rho}\right] &= -i\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right)\\
\left[P_\mu, J_{\nu\rho}\right] &= -i\left(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\right)\\
\left[J_{\mu\nu}, J_{\rho\sigma}\right] &= -i\left(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}\right)
\end{aligned}$$

and the rest vanish.

Solution. Recall that $P_\mu = -i\partial_\mu$, $D = -ix^\mu\partial_\mu$, $J_{\mu\nu} = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ and $K_\mu = -i(2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu)$. For the dilation and translation

$$\begin{aligned}
\left[D, P_\mu\right] &= -x^\nu\partial_\nu\partial_\mu + \partial_\mu x^\nu\partial_\nu\\
&\quad - x^\nu\partial_\nu\partial_\mu + \delta^\nu_\mu\partial_\mu + x^\nu\partial_\nu\partial_\mu\\
&= i(-i\partial_\mu) = iP_\mu. \qquad (1)
\end{aligned}$$

For the dilation and special conformal transformation

$$\begin{aligned}
\left[D, K_\mu\right] =& - x^\nu\partial_\nu(2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu) + (2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu)x^\nu\partial_\nu\\
=& - 2x_\mu x^\sigma\partial_\sigma - 2x_\mu(x\cdot\partial)(x\cdot\partial) + 2x^2\partial_\mu + x^\nu x^2\partial_\nu\partial_\mu + 2x_\mu(x\cdot\partial)(x\cdot\partial) - x^2\partial_\mu - x^2x^\nu\partial_\mu\partial_\nu\\
&= -2x_\mu x^\sigma\partial_\sigma + x^2\partial_\mu = -iK_\mu \qquad (2)
\end{aligned}$$

For the special conformal transformation and translation

$$\begin{aligned}
\left[K_\mu, P_\nu\right] =& - (2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu)\partial_\nu + \partial_\nu(2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu)\\
=& - 2x_\mu x^\sigma\partial_\sigma\partial_\nu + x^2\partial_\mu\partial_\nu + 2\eta_{\nu\mu}x^\sigma\partial_\sigma + 2x_\mu\partial_\nu + 2x_\mu x^\sigma\partial_\nu\partial_\sigma - 2x_\nu\partial_\mu - x^2\partial_\nu\partial_\mu\\
=&2\eta_{\nu\mu}x^\sigma\partial_\sigma + 2x_\mu\partial_\nu - 2x_\nu\partial_\mu\\
=&2i(\eta_{\mu\nu}D + J_{\mu\nu}) \qquad (3)
\end{aligned}$$

For the special conformal transformation and Lorentz rotation

$$\begin{aligned}
\left[K_\mu, J_{\nu\rho}\right] =& - (2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu)(x_\nu\partial_\rho - x_\rho\partial_\nu) + (x_\nu\partial_\rho - x_\rho\partial_\nu)(2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu)\\
=& - 2x_\mu x_\nu\partial_\rho - 2x_\mu x^\sigma x_\nu\partial_\sigma\partial_\rho + 2x_\mu x_\rho\partial_\nu + 2x_\mu x^\sigma x_\rho\partial_\sigma\partial_\nu\\
&+ x^2\eta_{\mu\nu}\partial_\rho + x^2x_\nu\partial_\mu\partial_\rho - x^2\eta_{\mu\rho}\partial_\nu - x^2x_\rho\partial_\mu\partial_\nu\\
&+ 2x_\nu\eta_{\rho\mu}x^\sigma\partial_\sigma + 2x_\nu x_\mu\partial_\rho + 2x_\nu x_\mu x^\sigma\partial_\rho\partial_\sigma - 2x_\nu x_\rho\partial_\mu - x_\nu x^2\partial_\rho\partial_\mu\\
&- 2x_\rho\eta_{\nu\mu}x^\sigma\partial_\sigma - 2x_\rho x_\mu\partial_\nu - x_\rho x_\mu x^\sigma\partial_\nu\partial_\sigma + 2x_\rho x_\nu\partial_\mu + x_\rho x^2\partial_\nu\partial_\mu\\
=& - \eta_{\mu\nu}(x_\rho x^\sigma\partial_\sigma - x^2\partial_\rho) + \eta_{\mu\rho}(x_\nu x^\sigma\partial_\sigma - x^2\partial_\nu)\\
=& - i\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right) \qquad (4)
\end{aligned}$$

For the translation and Lorentz rotation

$$\begin{aligned}
\left[P_\mu, J_{\nu\rho}\right] =& - \partial_\mu(x_\nu\partial_\rho - x_\rho\partial_\nu) + (x_\nu\partial_\rho - x_\rho\partial_\nu)\partial_\mu\\
=& - \eta_{\mu\nu}\partial_\rho - x_\nu\partial_\mu\partial_\rho + \eta_{\mu\rho}\partial_\nu + x_\rho\partial_\mu\partial_\nu + x_\nu\partial_\rho\partial_\mu - x_\rho\partial_\nu\partial_\mu\\
=& - \eta_{\mu\nu}\partial_\rho + \eta_{\mu\rho}\partial_\nu = -i\left(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\right) \qquad (5)
\end{aligned}$$

Problem 1 continued on next page. . .

For the last one,

$$\begin{aligned}\left[J_{\mu\nu}, J_{\rho\sigma}\right] &= (-i)^2 \left[x_\mu\partial_\nu - x_\nu\partial_\mu, x_\rho\partial_\sigma - x_\sigma\partial_\rho\right] \\ &= (-i)\left((-i)\left[x_\mu\partial_\nu, x_\rho\partial_\sigma\right] - (-i)\left[x_\mu\partial_\nu, x_\sigma\partial_\rho\right] - (-i)\left[x_\nu\partial_\mu, x_\rho\partial_\sigma\right] + (-i)\left[x_\nu\partial_\mu, x_\sigma\partial_\rho\right]\right)\end{aligned}$$

Since it has some symmetry, we only need to work out the first term

$$\begin{aligned}\left[x_\mu\partial_\nu, x_\rho\partial_\sigma\right] =& x_\mu\partial_\nu(x_\rho\partial_\sigma) - x_\rho\partial_\sigma(x_\mu\partial_\nu) \\ =& x_\mu\eta_{\nu\rho}\partial_\sigma + x_\mu x_\rho\partial_\nu\partial_\sigma - x_\rho\eta_{\sigma\mu}\partial_\nu - x_\rho x_\mu\partial_\sigma\partial_\nu \\ =& \left(\eta_{\nu\rho}x_\mu\partial_\sigma - \eta_{\sigma\mu}x_\rho\partial_\nu\right)\end{aligned} \tag{6}$$

by permuting the indices, we have

$$\begin{aligned}\left[x_\mu\partial_\nu, x_\rho\partial_\sigma\right] &= \eta_{\nu\rho}x_\mu\partial_\sigma - \eta_{\sigma\mu}x_\rho\partial_\nu \\ \left[x_\mu\partial_\nu, x_\sigma\partial_\rho\right] &= \eta_{\nu\sigma}x_\mu\partial_\rho - \eta_{\rho\mu}x_\sigma\partial_\nu \\ \left[x_\nu\partial_\mu, x_\rho\partial_\sigma\right] &= \eta_{\mu\rho}x_\nu\partial_\sigma - \eta_{\sigma\nu}x_\rho\partial_\mu \\ \left[x_\nu\partial_\mu, x_\sigma\partial_\rho\right] &= \eta_{\mu\sigma}x_\nu\partial_\rho - \eta_{\rho\nu}x_\sigma\partial_\mu\end{aligned}$$

Finally we get

$$\begin{aligned}\left[J_{\mu\nu}, J_{\rho\sigma}\right] =& -i\left(\eta_{\nu\rho}(-i)\left(x_\mu\partial_\sigma - x_\sigma\partial_\mu\right) - \eta_{\mu\rho}(-i)\left(x_\nu\partial_\sigma - x_\sigma\partial_\nu\right) - \eta_{\nu\sigma}(-i)\left(x_\mu\partial_\rho - x_\rho\partial_\mu\right) + \eta_{\mu\sigma}(-i)\left(x_\nu\partial_\rho - x_\rho\partial_\nu\right)\right) \\ =& -i\left(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\sigma}J_{\nu\rho}\right)\end{aligned} \tag{7}$$

Since the proof of other vanishing commutators are of completely the same philosophy, we won't repeat it here. □

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