Problem 1 Derive the OPE

$$T(z)X^\mu(w,\bar{w}) \sim \frac{1}{z-w}\partial X^\mu(w,\bar{w}) + \cdots$$

What does this imply for the conformal dimension of X^μ?

Solution. Recall that $T = -2 : \partial X \cdot \partial X :$, then we have

$$T(z)X^\mu(w,\bar{w}) = -2 : \partial X \cdot \partial X : X(w,\bar{w}).$$

The trick we are going to use is the Wick theorem, thus, we need to deal with the contraction $\langle \partial X^\mu(z) X^\nu(w)\rangle$. Notice that

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\frac{\eta^{\mu\nu}}{4}(\ln(z-w) + \ln(\bar{z}-\bar{w})),$$

from which we have

$$\langle \partial X^\mu(z) X^\nu(w)\rangle = -\frac{\eta^{\mu\nu}}{4(z-w)}.$$

Using this contraction, we have

$$\begin{aligned} T(z)X^\mu(w,\bar{w}) &= -2 : \partial X \cdot \partial X X(w,\bar{w}) : -4\partial X^\nu(z)\langle \partial X_\nu(z) X^\mu(w,\bar{w})\rangle \\ &= \frac{\partial X^\mu(z)}{z-w} + \cdots \end{aligned} \quad (1)$$

Notice that $\frac{\partial X^\mu(z)}{z-w} = \frac{\partial X^\mu(w,\bar{w})}{z-w} + \cdots$, we thus have $T(z)X^\mu(w,\bar{w}) \sim \frac{1}{z-w}\partial X^\mu(w,\bar{w}) + \cdots$. This means that the conformal dimension of X^μ is $h = 0$. □

Problem 2 (i) Use the result of the previous problem to deduce the OPE of $T(z)$ with each of the following operators:

$$\partial X^\mu(w,\bar{w}) \quad \bar{\partial} X^\mu(w,\bar{w}), \quad \partial^2 X^\mu(w,\bar{w})$$

(ii) What do these results imply for the conformal dimension $(h,\tilde{h})$ (if any) in each case?

Solution. (i) The result can be directly take the derivatives over w and $\bar{w}$ with $T(z)X^\mu(w,\bar{w}) \sim \frac{1}{z-w}\partial X^\mu(w,\bar{w}) + \cdots$. The results are

$$T(z)\partial X^\mu(w) = \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w} + \cdots$$

$$T(z)\bar{\partial} X^\mu(w) = \frac{\partial\bar{\partial} X^\mu(w,\bar{w})}{z-w} + \cdots$$

$$T(z)\partial^2 X^\mu(w) = \frac{2\partial X^\mu(w)}{(z-w)^3} + \frac{2\partial^2 X^\mu(w)}{(z-w)^2} + \frac{\partial^3 X^\mu(w)}{z-w} + \cdots$$

(ii)For an operator $\mathcal{O}$ with conformal dimension $(h,\tilde{h})$, the OPE with T and $\bar{T}$ takes the form

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= \ldots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots \\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \ldots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \ldots \end{aligned}$$

To determine, we also need to calculate the following terms

$$\tilde{T}(\bar{z})X^\mu(w,\bar{w}) = \frac{\bar{\partial} X^\mu(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial^2 X^\mu(\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

By taking derivatives, we obtain the similar result as the above, but we won's do it here.

Thus from the result of (i), we can only infer the conformal dimesnion h of each field. For three fields $\partial X^\mu(w,\bar{w}) \quad \bar{\partial} X^\mu(w,\bar{w}), \quad \partial^2 X^\mu(w,\bar{w})$, they are respectively 1, 0 and 2.

Problem 3 Show that

$$\left[\alpha_m^\mu, \alpha_n^\nu\right] = \left[\widetilde{\alpha}_m^\mu, \widetilde{\alpha}_n^\nu\right] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad \left[\alpha_m^\mu, \widetilde{\alpha}_n^\nu\right] = 0$$

by using the OPE of the field $\partial X^\mu(z,\bar{z})$ with itself and with $\bar{\partial} X^\mu(z,\bar{z})$

Solution. Recall that the derivatives of $X^\mu(z)$ are

$$\partial X^\mu(z,\bar{z}) = -\frac{i}{2}\sum_n \alpha_n^\mu z^{-n-1}, \quad \bar{\partial} X^\mu(z,\bar{z}) = -\frac{i}{2}\sum_n \tilde{\alpha}_n^\mu \bar{z}^{-n-1}$$

it follows that

$$\alpha_m^\mu = \frac{1}{\pi}\oint dz z^m \partial X^\mu(z,\bar{z}), \quad \tilde{\alpha}_m^\mu = \frac{1}{\pi}\oint d\bar{z}\bar{z}^m \bar{\partial} X^\mu(z,\bar{z}),$$

where integration is performed over some contour around $z=0$ pole point.

Thus we have

$$\left[\alpha_m^\mu, \alpha_n^\nu\right] = \frac{1}{\pi^2}\oint\oint dz dw z^m w^n \left(\partial X^\mu(z,\bar{z})\partial X^\nu(w,\bar{w}) - \partial X^\nu(w,\bar{w})\partial X^\mu(z,\bar{z})\right) = m\eta^{\mu\nu}\delta_{m+n,0}.$$

Note that here we have used the result

$$X^\mu(z,\bar{z})X^\nu(w,\bar{w}) = -\frac{\eta^{\mu\nu}}{4}(\ln(z-w) + \ln(\bar{z}-\bar{w})). \tag{2}$$

Doing the same kind of calculation, we also have $\left[\widetilde{\alpha}_m^\mu, \widetilde{\alpha}_n^\nu\right] = m\eta^{\mu\nu}\delta_{m+n,0}$.

For the case of $\left[\alpha_m^\mu, \widetilde{\alpha}_n^\nu\right] = 0$, taking derivative with z and $\bar{w}$ of Eq. (2), we have

$$\partial X^\mu(z,\bar{z})\bar{\partial} X^\nu(w,\bar{w}) = 0. \tag{3}$$

Therefore, we have

$$\left[\alpha_m^\mu, \widetilde{\alpha}_n^\nu\right] = \frac{1}{\pi^2}\oint\oint dz d\bar{w} z^m \bar{w}^n \left(\partial X^\mu(z,\bar{z})\bar{\partial} X^\nu(w,\bar{w}) - \bar{\partial} X^\nu(w,\bar{w})\partial X^\mu(z,\bar{z})\right) = 0 \tag{4}$$

□

Problem 4 Consider a conformal field $\Phi(z)$ of dimension h and a mode expansion of the form

$$\Phi(z) = \sum_{n\in\mathbb{Z}} \frac{\Phi_n}{z^{n+h}}. \tag{5}$$

Using contour-integral methods, like those of Exercise 3.2, evaluate the commutator $[L_m, \Phi_n]$.

Solution. Recall that from the energy-momentum tensor we have

$$L_m = \frac{1}{2\pi i}\oint dw T(w) w^{m+1}$$

Thus we have the following result

$$[L_m, \Phi_n] = \frac{1}{2\pi i}\oint dz z^{n+h-1}\frac{1}{2\pi i}\oint_\Gamma dw T(w)\Phi(z) w^{m+1}$$

where Γ is some contour of w around z.

Recall that OPE for $T(z)$ and $\Phi(z,\bar{z})$ is $T(z)\Phi(w,\bar{w}) = \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{z-w}\partial\Phi(w,\bar{w}) + \ldots$, using this formula to do the integration, we obtain

$$[L_m, \Phi_n] = \frac{1}{2\pi i}\oint dz z^{n+h-1}\left(h(m+1)z^m\Phi(z) + z^{m+1}\partial\Phi(z)\right) \tag{6}$$

From the Laurent expansion

$$\Phi(z) = \sum_{n\in\mathbb{Z}} \frac{\Phi_n}{z^{n+h}}. \tag{7}$$

we see that

$$\Phi_n = \frac{1}{2\pi i}\oint dz\Phi(z)z^{n+h-1}, \quad \frac{1}{2\pi i}\oint dz\partial\Phi(z)z^{n+h} = -(n+h)\Phi_n$$

Substituting the result into Eq. (6), we obtain

$$\frac{1}{2\pi i}\oint dz z^{n+h-1}\left(h(m+1)z^m\Phi(z) + z^{m+1}\partial\Phi(z)\right) = -(n+m(1-h))\Phi_{n+m} \tag{8}$$

Thus $[L_m, \Phi_n] = -(n+m(1-h))\Phi_{n+m}$. □

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