Problem 1 Drive the commutation relations

$$[J^A_m, J^B_n] = \frac{1}{2}km\delta^{AB}\delta_{m+n,0} + if^{AB}_C J^C_{m+n}$$

for generators of Kac-Moody algebra

Solution. Since $J^A(z) = \sum_{m\in\mathbb{Z}} \frac{J^A_m}{z^{m+1}}$ and $J^B(w) = \sum_{n\in\mathbb{Z}} \frac{J^B_n}{w^{n+1}}$, we have

$$J^A_m = \frac{1}{2\pi i}\oint J^A(z) z^m dz, \quad J^B_n = \frac{1}{2\pi i}\oint J^B(w) w^n dw. \tag{1}$$

Recall the OPE

$$\mathsf{R}\{J^A(z)J^B(w)\} \sim \frac{k\delta^{AB}}{2(z-w)^2} + \frac{if^{AB}_C J^C(w)}{z-w} + \dots, \tag{2}$$

where, for clarity, we have stress the radial ordering operator, which we denote as R. These results imply that

$$\begin{aligned}
[J^A_m, J^B_n] &= \frac{1}{2\pi i}\oint dz \frac{1}{2\pi i}\oint dw (J^A(z)z^m J^B(w)w^n - J^B(w)w^n J^A(z)z^m) \\
&= \frac{1}{2\pi i}\oint dz \frac{1}{2\pi i}\oint dw z^m w^n \mathsf{R}\{J^A(z)z^m J^B(w)\} \qquad (3)\\
&= \frac{1}{2\pi i}\oint dz \frac{1}{2\pi i}\oint dw z^m w^n (\frac{k\delta^{AB}}{2(z-w)^2} + \frac{if^{AB}_C J^C(w)}{z-w} + \dots) \\
&= \frac{1}{2\pi i}\oint dz \frac{1}{2\pi i}\oint dw z^m w^n (\frac{k\delta^{AB}}{2(z-w)^2} + \frac{if^{AB}_C \sum_{l\in\mathbb{Z}} J^C_l z^{-(l+1)}}{z-w} + \dots) \\
&= \frac{1}{2}km\delta^{AB}\delta_{m+n,0} + if^{AB}_C J^C_{m+n}. \qquad (4)
\end{aligned}$$

Note that, here, the integration of z is over the contour around $z = w$, and the integration of w is over the contour around $w = 0$. □

Problem 2 Derive the central charge $c = \frac{k \dim G}{k+\tilde{h}_G}$ for the energy-momentum tensor $T(z) = \frac{1}{k+\tilde{h}_G}\sum_{A=1}^{\dim G} :J^A(z)J^A(z):$.

Solution. Since $J^A(z) = \sum_{m\in\mathbb{Z}} \frac{J^A_m}{z^{m+1}}$, we have

$$J^A_m = \frac{1}{2\pi i}\oint J^A(z) z^m dz. \tag{5}$$

By definition, we have

$$L_n = \frac{1}{2\pi i}\oint z^{n+1} T(z) dz \tag{6}$$

From the expression of $T(z)$ and mode expansion of $J^A(z)$ we have

$$L_n = \frac{1}{2(k+\tilde{h}_G)}\sum_{A=1}^{\dim G}\left(\sum_{l\leq -1} J^A_l J^A_{n-l} + \sum_{l>-1} J^A_{n-l} J^A_l\right). \tag{7}$$

Using the OPE of energy-momentum tensor with itself,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

we obtain that

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n,0} \tag{8}$$

Problem 2 continued on next page...

Then consider the expectation of L_2L_{-2} over vacuum state $|0\rangle$, and using the fact $L_{-2}|0\rangle = \frac{1}{2(k+\tilde{h}_G)}\sum_{A=1}^{\dim G} J_{-1}^A J_{-1}^A|0\rangle$, we have

$$\frac{c}{2} = \langle 0|L_2L_{-2}|0\rangle = (\frac{1}{2(k+\tilde{h}_G)})^2 \sum_{A,B} \left\langle 0 \left| J_1^B j_1^B J_{-1}^A J_{-1}^A \right| 0 \right\rangle \tag{9}$$

$$= (\frac{1}{2(k+\tilde{h}_G)})^2 \sum_{A,B} \left(\left\langle 0 \left| J_1^B \left[J_1^B, J_{-1}^A\right] J_{-1}^A \right| 0 \right\rangle + \left\langle 0 \left| \left[J_1^B, J_{-1}^A\right] \left[J_1^B, J_{-1}^A\right] \right| 0 \right\rangle \right) \tag{10}$$

$$= (\frac{1}{2(k+\tilde{h}_G)})^2 \sum_{A,B} \left(i \sum_C f^{BAC} \left\langle 0 \left| J_1^B \left[J_0^B, J_{-1}^A\right] \right| 0 \right\rangle + k \left\langle 0 \left| J_1^B J_{-1}^A \right| 0 \right\rangle \delta^{BA} + k^2 \delta^{BA} \right) \tag{11}$$

$$= (\frac{1}{2(k+\tilde{h}_G)})^2 \sum_{A,B} \left(- \sum_{C,D} f^{BAC} f^{CAD} \left\langle 0 \left| \left[J_1^B, J_{-1}^D\right] \right| 0 \right\rangle + 2k^2 \delta^{BA} \right) \tag{12}$$

$$= (\frac{1}{2(k+\tilde{h}_G)})^2 \sum_{A,B} \left(- \sum_{C,D} f^{BAC} f^{CAD} k \delta^{BD} + 2k^2 \delta^{BA} \right) \tag{13}$$

$$= (\frac{1}{2(k+\tilde{h}_G)})^2 \left(\sum_D 2\tilde{h}_G k + \sum_A 2k^2 \right) = \frac{k \dim G}{2\left(k+\tilde{h}_G\right)} \tag{14}$$

This implies that $c = \frac{k \dim G}{k+\tilde{h}_G}$. □

Problem 3 Show that the central charge is

$$c_{bc}(\epsilon, \lambda) = -2\epsilon(6\lambda^2 - 6\lambda + 1)$$

for the ghost field energy-momentum tensor

$$T_{bc}(z) = -\lambda : b(z)\partial c(z) : -\epsilon(1-\lambda) : c(z)\partial b(z) :$$

Solution. Firstly, recall that $c(z)b(w) \sim \frac{1}{z-w}$ and $b(z)c(w)\frac{\epsilon}{z-w}$, we have

$$b(z)\partial c(w) \sim \frac{\epsilon}{(z-w)^2} \tag{15}$$

$$\partial b(z)c(w) \sim -\frac{\epsilon}{(z-w)^2} \tag{16}$$

$$c(z)\partial b(w) \sim \frac{1}{(z-w)^2} \tag{17}$$

$$\partial c(z)b(w) \sim -\frac{1}{(z-w)^2} \tag{18}$$

$$\partial c(z)\partial b(w) \sim -\frac{2}{(z-w)^3} \tag{19}$$

$$\partial b(z)\partial c(w) \sim -\frac{2\epsilon}{(z-w)^3} \tag{20}$$

Using the OPE of energy-momentum tensor with itself,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots,$$

we see that to determine c, only $(z-w)^{-4}$ term are of interest.

Problem 3 continued on next page. . .

To calculate $T_{bc}(z)T_{bc}(w)$, it's sufficient for us to work us the following terms

$$:b(z)\partial c(z)::b(w)\partial c(w):=-\epsilon(z-w)^{-4}+\text{regular terms}+\text{other singular terms} \tag{21}$$
$$:b(z)\partial c(z)::c(w)\partial b(w):=-2\epsilon(z-w)^{-4}+\text{regular terms}+\text{other singular terms} \tag{22}$$
$$:c(z)\partial b(z)::b(w)\partial c(w):=-2\epsilon(z-w)^{-4}+\text{regular terms}+\text{other singular terms} \tag{23}$$
$$:c(z)\partial b(z)::c(w)\partial b(w):=-\epsilon(z-w)^{-4}+\text{regular terms}+\text{other singular terms} \tag{24}$$
$$\tag{25}$$

Combining these results and the expression of T_{bc}, we have $c_{bc}=-2\epsilon(6\lambda^2-6\lambda+1)$. □

Problem 4 Derive the following OPE

$$\begin{aligned} T(z)j_B(w) =&\frac{1}{(z-w)^2}j_B(w)+\frac{1}{z-w}\partial_w j_B(w)+\frac{1}{(z-w)^3}\partial_w c(w)\\ &+\frac{\frac{D}{2}-4}{(z-w)^4}c(w) \end{aligned} \tag{26}$$

Prove that by adding a term $3/2\partial^2 c(z)$ to the above j_B, the new BRST current has conformal dimension $(1,0)$ at critical dimension $D=26$.

Solution. Recall that

$$T(z)=T_X(z)+T_{bc}(z)$$

where $T_X(z)=-2:\partial X\cdot\partial X:$ and $T_{bc}(z)=-2:b(z)\partial c(z):+:c(z)\partial b(z):$, and the current is

$$j_B(z)=:c(z)\left(T_X(z)+\frac{1}{2}T_{bc}(z)\right):. \tag{27}$$

Then, we have

$$\begin{aligned} T(z)j_B(w)\sim&T_X(z)c(w)T_X(w)+T_{bc}(z)c(w)\left(T_X(w)+\frac{1}{2}T_{bc}(w)\right)\\ \sim&c(w)\left(\frac{D/2}{(z-w)^4}+\frac{2T_X(w)}{(z-w)^2}+\frac{\partial T_X(w)}{z-w}\right)\\ &+T_X(w)\left(-\frac{c(w)}{(z-w)^2}+\frac{\partial c(w)}{(z-w)}\right)+\frac{1}{2}T_{bc}(z)c(w)T_{bc}(w) \end{aligned} \tag{28}$$

where

$$\begin{aligned} \frac{1}{2}T_{bc}(z)c(w)T_{bc}(w)=&[c(z)(\partial b(z))-2b(z)\partial c(z)][c(w)(c(w)\partial b(w)-2b(w)\partial c(w))b(w)]\\ \sim&\left(\partial_z\frac{1}{z-w}\right)\frac{1}{z-w}\partial c(w)-\left(\partial_z\frac{1}{z-w}\right)\left(\partial_w\frac{1}{z-w}\right)c(w)\\ &-2\partial_z\left(\frac{1}{(z-w)^2}\right)\partial c(w)+2\partial_z\left(\frac{1}{z-w}\partial_w\frac{1}{z-w}\right)c(w)\\ &+\frac{1}{2}\left(\frac{cT_{bc}(w)}{(z-w)^2}+\frac{\partial\left(cT_{bc}\right)(w)}{z-w}\right) \end{aligned} \tag{29}$$

Note that we have used the result

$$T_{bc}(z)c(w)\sim\frac{-c(w)}{(z-w)^2}+\frac{\partial_w c(w)}{(z-w)} \tag{30}$$

Problem 4 continued on next page. . .

and

$$T_{bc}(z)b(w) \sim 2b(z)\partial_z(\frac{1}{z-w}) + \frac{\partial_z b(z)}{z-w} \tag{31}$$

Substituting Eq. (29) to Eq. (28), we obtain that

$$T(z)j_B(w) = \frac{1}{(z-w)^2}j_B(w) + \frac{1}{z-w}\partial_w j_B(w) + \frac{1}{(z-w)^3}\partial_w c(w) \tag{32}$$

$$+ \frac{\frac{D}{2}-4}{(z-w)^4}c(w) \tag{33}$$

Now consider the new BRST current

$$\begin{aligned} j_B(z) &= c(z)T_X(z) + \frac{1}{2} : c(z)T^{gh}(z) : + \frac{3}{2}\partial^2 c(z) \\ &= c(z)T_X(z) + : b(z)c(z)\partial c(z) : + \frac{3}{2}\partial^2 c(z) \end{aligned} \tag{34}$$

We have the OPE

$$T(z)j_B(w) \sim \frac{D-26}{2(z-w)^4}c(w) + \frac{1}{(z-w)^2}j_B(w) + \frac{1}{z-w}\partial j_B(w) \tag{35}$$

We see that when $D = 26$, we have

$$T(z)j_B(w) \sim \frac{1}{(z-w)^2}j_B(w) + \frac{1}{z-w}\partial j_B(w) \tag{36}$$

Recall that for field with conformal dimension $(h, \tilde{h})$ we have

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= \ldots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots \\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \ldots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \ldots \end{aligned} \tag{37}$$

Which means that for j_b, the conformal dimension $h = 1$. □

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