Problem 1 Drive the OPE for

$$j_B(z)j_B(w) \sim -\frac{c^X-18}{2(z-w)^3} :c(w)\partial c(w): -\frac{c^X-18}{4(z-w)^2} :c(w)\partial^2 c(w): -\frac{c^X-26}{12(z-w)} :c(w)\partial^3 c(w): \tag{1}$$

Solution. Recall the definitions

$$j_B(z) =: c(z)T_X(z) : +\frac{1}{2} : c(z)T_{bc}(z) : +\frac{3}{2}\partial^2 c(z), \tag{2}$$

$$T^{bc}(z) = -2 : (b\partial c)(z) : + : (c\partial b)(z) :, \tag{3}$$

and the following OPE's

$$T_X(z)T_X(w) = \frac{c^X/2}{(z-w)^4} + \frac{2T_X(w)}{(z-w)^2} + \frac{\partial T_X(w)}{z-w} \tag{4}$$

$$c(z)b(w) = 1/(z-w), \quad b(z)c(w) = 1/(z-w). \tag{5}$$

$$T_{bc}(z)T_{bc}(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T_{bc}(w) + \frac{1}{z-w}\partial T_{bc}(w) + \cdots \tag{6}$$

here we've taken $\lambda = 2$ and $\epsilon = 1$, thus $c = -26$

$$T_{bc}(z)c(w) = \frac{-c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w} \tag{7}$$

$$T_{bc}(z)b(w) = \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} \tag{8}$$

$$cT_{bc} = c[(\partial b)c - 2b\partial(c)] = 2c(\partial c)b \tag{9}$$

Using the results, we have (hereinafter, to avoid cluttering of equations, we will omitted normal order symbol : :, whenever it's possible to make ambiguity, we will write it explicitly)

$$\begin{aligned} j_B(z)j_B(w) =& j_B(z)[c(w)T_X(w) + \frac{1}{2}c(w)T_{bc}(w) + \frac{3}{2}\partial^2 c(w)] \\ =& j_B(z)[c(w)T_X(w) + c(w)(\partial c(w))b(w) + \frac{3}{2}\partial^2 c(w)] \end{aligned} \tag{10}$$

To calculate this OPE, let us first calculate some useful OPE's.

$$j_B(z)c(w) = \left(cT_X + bc\partial c + \frac{3}{2}\partial^2 c\right)(z)c(w) \sim bc\partial c(z)c(w) = \frac{c\partial c(z)}{z-w} \sim \frac{c\partial c(w)}{z-w} \tag{11}$$

Note here in the last step we have used Taylor expansion of $c\partial c(z)$ around $z = w$.

$$\begin{aligned} j_B(z)b(w) &\sim \frac{T_X(z)}{z-w} - \frac{b\partial c(z)}{z-w} + bc(z)\partial_z\frac{1}{z-w} + \frac{3}{2}\partial_z^2\frac{1}{z-w} \\ &\sim \frac{T_X(z)}{z-w} - \frac{b\partial c(z)}{z-w} - \frac{bc(z)}{(z-w)^2} + \frac{3}{(z-w)^3} \\ &\sim \frac{T_X(w)+\cdots}{z-w} - \frac{b\partial c(w)+\cdots}{z-w} - \frac{bc(w)+(z-w)\partial(bc)(w)+\cdots}{(z-w)^2} + \frac{3}{(z-w)^3} \\ &\sim \frac{3}{(z-w)^3} - \frac{bc(w)}{(z-w)^2} + \frac{T_X(w)-(\partial b)c(w)-2b\partial c(w)}{z-w} \\ &= \frac{3}{(z-w)^3} - \frac{bc(w)}{(z-w)^2} + \frac{T(w)}{z-w} \end{aligned} \tag{12}$$

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Here $T = T_X + T_{bc}$.

The last one we need is

$$\begin{aligned}
&j_B(z)T_X(w) = \left(cT_X + bc\partial c + \frac{3}{2}\partial^2 c\right)(z)T_X(w) \\
=&c(z)T_X(z)T_X(w) = c(z)(\frac{c^X/2}{(z-w)^4} + \frac{2T_X(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}) \\
=&(c(w) + (z-w)\partial c(w) + (z-w)^2\partial^2 c(w)/2 + (z-w)^3\partial^3 c(w)/6)(\frac{c^X/2}{(z-w)^4} + \frac{2T_X(w)}{(z-w)^2} + \frac{\partial T_X(w)}{z-w})
\end{aligned} \tag{13}$$

Notice here we have repeatedly used the fact that the contraction of matter field and ghost field vanishes.

With these preparations, we are now at a position to calculate the OPE of BRST currents. Using the contraction formula for $j_B(z)$ and $c(w)$ and for $j_B(z)$ and $T_X(w)$, we have

$$\begin{aligned}
&j_B(z)c(w)T_X(w) \\
=&\frac{c\partial c(w)}{z-w}T_X(w) \\
&+ c(w)(c(w) + (z-w)\partial c(w) + (z-w)^2\partial^2 c(w)/2 + (z-w)^3\partial^3 c(w)/6)(\frac{c^X/2}{(z-w)^4} + \frac{2T_X(w)}{(z-w)^2} + \frac{\partial T_X(w)}{z-w})
\end{aligned}$$

Using the contraction formula for $j_B(z)$ and $c(w)$ and for $j_B(z)$ and $b(w)$, we have

$$\begin{aligned}
j_B(z)c(w)(\partial c(w))b(w) =&\frac{c\partial c(w)}{z-w} : (\partial c(w))b(w) : + : c(w)b(w) : \partial_w(\frac{c\partial c(w)}{z-w}) \\
&+ : c(w)(\partial c(w)) : (\frac{3}{(z-w)^3} - \frac{bc(w)}{(z-w)^2} + \frac{T(w)}{z-w})
\end{aligned} \tag{14}$$

Using the contraction formula for $j_B(z)$ and $c(w)$, we have

$$j_B(z)\frac{3}{2}\partial^2 c(w) = \frac{3}{2}\partial^2(\frac{c\partial c(w)}{z-w}) \tag{15}$$

Combining the above three equation together, we obtain what we want

$$j_B(z)j_B(w) \sim -\frac{c^X - 18}{2(z-w)^3} : c(w)\partial c(w) : -\frac{c^X - 18}{4(z-w)^2} : c(w)\partial^2 c(w) : -\frac{c^X - 26}{12(z-w)} : c(w)\partial^3 c(w) : \tag{16}$$

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