Problem 1 Considcr a massless supcrsymmctric particle (or supcrparticle) propagating in D -dimensional Minkowski space-time. It is described by D bosonic fields $X^\mu(\tau)$ and D Majorana fermions $\psi^\mu(\tau)$. The action is

$$S_0 = \int d\tau \left(\frac{1}{2} \dot{X}^\mu \dot{X}_\mu - i\psi^\mu \dot{\psi}_\mu \right)$$

(i) Derive the field equations for X^μ, ψ^μ.

(ii) Show that the action is invariant under the global supersymmetry transformations

$$\delta X^\mu = i\varepsilon\psi^\mu, \quad \delta\psi^\mu = \frac{1}{2}\varepsilon\dot{X}^\mu$$

where ε is an infinitesimal real constant Grassmann parameter.

(iii) Suppose that δ_1 and δ_2 are two infinitesimal supersymmetry transformations with parameters ε_1 and ε_2, respectively. Show that the commutator $[\delta_1, \delta_2]$ gives a τ translation by an amount $\delta\tau$. Determine $\delta\tau$ and explain why $\delta\tau$ is real.

Solution.

(i) The Lagrangian is $L = \frac{1}{2}\dot{X}^\mu \dot{X}_\mu - i\psi^\mu\dot{\psi}_\mu$, using the Euler-Lagrange equation for X^μ, we have the equation of motion

$$\ddot{X}^\mu = 0. \tag{1}$$

Similarly, for fermionic part, we have

$$\frac{\partial}{\partial \psi^\nu} \psi^\mu \dot{\psi}_\mu = \dot{\psi}_\nu \tag{2}$$

and

$$\frac{\partial}{\partial \dot{\psi}^\nu} \psi^\mu \dot{\psi}_\mu = -\psi^\mu \frac{\partial}{\partial \dot{\psi}^\nu} \dot{\psi}_\mu = -\psi_\nu \tag{3}$$

Then using the Euler-Lagrange equation, we obtain the equation of motion as

$$\dot{\psi}_\nu = 0. \tag{4}$$

(ii)The symmetry transformation is

$$\begin{cases} \delta X^\mu = i\varepsilon\psi^\mu, & \delta\psi^\mu = \frac{1}{2}\varepsilon\dot{X}^\mu \\ \delta \dot{X}^\mu = i\varepsilon\dot{\psi}^\mu, & \delta\dot{\psi}^\mu = \frac{1}{2}\varepsilon\ddot{X}^\mu \end{cases} \tag{5}$$

The variation of the action is

$$\begin{aligned} \delta S_0 &= \int d\tau (\eta_{\mu\nu} i\varepsilon\dot{\psi}^\mu \dot{X}^\nu - i\frac{1}{2}\varepsilon\dot{X}^\mu\dot{\psi}^\nu\eta_{\mu\nu} - i\psi^\nu\frac{1}{2}\varepsilon\ddot{X}^\mu\eta_{\mu\nu}) \\ &= \int d\tau (\eta_{\mu\nu} i\varepsilon\dot{X}^\mu\dot{\psi}^\nu - i\frac{1}{2}\varepsilon\dot{X}^\mu\dot{\psi}^\nu\eta_{\mu\nu} + i\frac{1}{2}\varepsilon\ddot{X}^\mu\dot{\psi}^\nu\eta_{\mu\nu}) \\ &= i\frac{\varepsilon}{2}\int d\tau \frac{d}{d\tau}(\dot{X}^\mu\psi^\nu\eta_{\mu\nu}) \end{aligned} \tag{6}$$

Under the condition that $\dot{X}^\mu\psi^\nu\eta_{\mu\nu}\Big|_{\tau_i}^{\tau_f} = 0$ (this is usually chosen as $\dot{X}^\mu\psi^\nu\eta_{\mu\nu} = 0$ at τ_i and τ_f), the action is invariant under the symmetry transformation.

(iii) Since we have the supersymmetric transformation

$$\delta_j X^\mu = i\varepsilon_j\psi^\mu, \quad \delta_j\psi^\mu = \frac{1}{2}\varepsilon_j\dot{X}^\mu \tag{7}$$

for $j = 1, 2$. Now let us calculate the action $[\delta_1, \delta_2]$ over the bosonic and fermionic fields

$$(\delta_1\delta_2 - \delta_2\delta_1)\, X^\mu = \frac{i}{2}(\varepsilon_2\varepsilon_1 - \varepsilon_1\varepsilon_2)\,\dot{X}^\mu \circeq \delta\tau\dot{X}^\mu \tag{8}$$

$$(\delta_1\delta_2 - \delta_2\delta_1)\, \psi^\mu = \frac{i}{2}(\varepsilon_2\varepsilon_1 - \varepsilon_1\varepsilon_2)\,\dot{\psi}^\mu \circeq \delta\tau\dot{\psi}^\mu \tag{9}$$

Problem 1 continued on next page...

Note that here we have used the anti-commuting relations between ε_1 and ε_2, and using $\hat{=}$ indicate the the relations that $[\delta_1, \delta_2]$ is actually the τ traslation. We see that

$$\delta\tau = \frac{i}{2}(\varepsilon_2\varepsilon_1 - \varepsilon_1\varepsilon_2). \tag{10}$$

Since ψ is Majorana field, it is real. From the variation of bosonic field $\delta_j X^\mu = i\varepsilon_j\psi^\mu$, we see that ε_j must be imaginary number, thus we conclude that $\delta\tau$ is real.

□

Problem 2 In Problem 4.1, supersymmetry was only a global symmetry, as ε did not depend on τ. To construct an action in which this symmetry is local, one needs to include the auxiliary field e and its fermionic partner, which we denote by χ. The action takes the form

$$\widetilde{S}_0 = \int d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{2e} + \frac{i\dot{X}^\mu \psi_\mu \chi}{e} - i\psi^\mu \dot{\psi}_\mu\right)$$

(i) Show that this action is reparametrization invariant, that is, it is invariant under the following infinitesimal transformations with parameter $\xi(\tau)$:

$$\begin{aligned} \delta X^\mu &= \xi \dot{X}^\mu, \quad \delta\psi^\mu = \xi\dot{\psi}^\mu \\ \delta e &= \tfrac{d}{d\tau}(\xi e), \quad \delta\chi = \tfrac{d}{d\tau}(\xi\chi) \end{aligned}$$

(ii) Show explicitly that the action is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta X^\mu = i\varepsilon\psi^\mu, \delta\psi^\mu &= \frac{1}{2e}\left(\dot{X}^\mu - i\chi\psi^\mu\right)\varepsilon \\ \delta\chi = \dot{\varepsilon}, \delta e &= -i\chi\varepsilon \end{aligned}$$

(iii) Show that in the gauge $e = 1$ and $\chi = 0$, one recovers the action in Problem 4.1 and the constraint equations $\dot{X}^2 = 0, \dot{X}\cdot\psi = 0$

Solution.

(i) Consider the infinitesimal transformation $\tau \to \tau - \xi(\tau)$, we have

$$\begin{aligned} \delta X^\mu &= \xi\dot{X}^\mu, & \delta\psi^\mu &= \xi\dot{\psi}^\mu \\ \delta\dot{X}^\mu &= \dot{\xi}\dot{X}^\mu + \xi\ddot{X}^\mu, & \delta\dot{\psi}^\mu &= \dot{\xi}\dot{\psi}^\mu + \xi\ddot{\psi}^\mu \\ \delta e &= \tfrac{d}{d\tau}(\xi e), & \delta\chi &= \tfrac{d}{d\tau}(\xi\chi) \end{aligned}$$

Thus the variation of the action is

$$\begin{aligned} \delta\tilde{S}_0 &= \int d\tau \left(\frac{\delta\dot{X}^\mu \dot{X}_\mu}{e} - \frac{\dot{X}^\mu\dot{X}_\mu}{2e^2}\delta e + \frac{i(\delta\dot{X}^\mu)\psi_\mu\chi}{e} + \frac{i\dot{X}^\mu(\delta\psi_\mu)\chi}{e} + \frac{i\dot{X}^\mu\psi_\mu\delta\chi}{e} - \frac{i\dot{X}^\mu\psi_\mu\chi}{e^2}\delta e - i(\delta\psi^\mu)\dot{\psi}_\mu - i\psi_\mu\delta\dot{\psi}^\mu\right) \\ &= \int d\tau \frac{d}{d\tau}\left(\xi\frac{\dot{X}^\mu\dot{X}_\mu}{2e} + \frac{i\xi\dot{X}^\mu\psi_\mu\chi}{e} - i\xi\psi^\mu\dot{\psi}_\mu\right) \end{aligned} \tag{11}$$

which is an integration of total derivative, thus vanishes. The action is invariant under reparameterization.

(ii) The variation of the action is

$$\begin{aligned} \delta\tilde{S}_0 &= \int d\tau \left(\frac{\delta\dot{X}^\mu \dot{X}_\mu}{e} - \frac{\dot{X}^\mu\dot{X}_\mu}{2e^2}\delta e + \frac{i(\delta\dot{X}^\mu)\psi_\mu\chi}{e} + \frac{i\dot{X}^\mu(\delta\psi_\mu)\chi}{e} + \frac{i\dot{X}^\mu\psi_\mu\delta\chi}{e} - \frac{i\dot{X}^\mu\psi_\mu\chi}{e^2}\delta e - i(\delta\psi^\mu)\dot{\psi}_\mu - i\psi_\mu\delta\dot{\psi}^\mu\right) \\ &= \int d\tau \frac{d}{d\tau}\left(\frac{i}{2e}\varepsilon\psi_\mu\dot{X}^\mu\right) \end{aligned} \tag{12}$$

Problem 2 continued on next page. . .

We have repeatedly used the anti-commuting relation of Grassmann numbers here. Since the variation is an integration of total derivative, thus vanishes. The action is invariant under the supersymmetry transformation.

(iii) The equation of motion for field χ is obtained from $\delta\tilde{S}_0/\delta\chi = 0$, which is obviously

$$\dot{X}\cdot\psi/e = 0.$$

Similarly, for e, the equation of motion is

$$\frac{\dot{X}^\mu\dot{X}_\mu}{2e^2} + \frac{\dot{X}^\mu\psi_\mu\chi}{e^2} = 0.$$

In the gauge $e = 1$ and $chi = 0$, we have the equation of motion $\dot{X}^2 = 0$, $\dot{X}\cdot\psi = 0$, and the action becomes

$$S_0 = \int d\tau\left(\frac{1}{2}\dot{X}^\mu\dot{X}_\mu - i\psi^\mu\dot{\psi}_\mu\right)$$

□

Problem 3 Consider quantization of the superparticle action in Problem 4.1

(i) Show that canonical quantization gives the equal-τ commutation and anticommutation relations

$$\left[X^\mu, \dot{X}^\nu\right] = i\eta^{\mu\nu} \quad \text{and} \quad \left\{\psi^\mu, \psi^\nu\right\} = \eta^{\mu\nu}$$

(ii) Explain why this describes a space-time fermion.

(iii) What is the significance of the constraints $\dot{X}^2 = 0$ and $\dot{X}\cdot\psi = 0$ obtained in Problem 4.2?

Solution.

(i) To do canonical quantization, we first consider the equation of motion

$$\ddot{X}^\mu = 0, \tag{13}$$

the general solution (mode expansion) is $X^\mu = x^\mu + p^\mu\tau$.

Similarly, for fermions, we have the equation of motion $\dot{\psi}^\mu = 0$, the general solution is $\psi^\mu(\tau) \equiv f^\mu$, here f^μ are D Majorana fermions.

The canonical quantization for bosonic and fermion modes are

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad \{f^\mu, f^\nu\} = \eta^{\mu\nu}. \tag{14}$$

all other commutators vanish. Now the fermionic anti-commutator $\left\{\psi^\mu, \psi^\nu\right\} = \eta^{\mu\nu}$ is obvious. For the bosonic part, recall that $\dot{X}^\mu = p^\mu$, thus we have $\left[X^\mu, \dot{X}^\nu\right] = i\eta^{\mu\nu}$.

(ii) To answer this question, we need to analyze the physical states. It's obvious that there are two parameters to label a quantum state $|k; f\rangle$ where k is the momentum and f for fermionic quantum number. The vacuum state is the zero-momentum state $p^\mu|0; f\rangle = 0$. But there are still fermionic freedom, which means that the vacuum is the space-time fermions. Similarly, for excited state $p^\mu|k^\mu; f\rangle = k^\mu|k^\mu; f\rangle$, which is the non-zero momentum states, since the existence of the fermionic quantum number, they are also spacetime fermions.

(iii) Recall that $\dot{X}^\mu = p^\mu$, and relativistic energy-mass equation $p^2 = \dot{X}^2 = m^2$, we see that the constraint $\dot{X}^2 = 0$ means that the spacetime fermions are massless.

To analyze the constraint $\dot{X}\cdot\psi = 0$, let us introduce the Dirac matrices $\gamma^\mu = \frac{1}{\sqrt{2}}f^\mu$, from the commutation relation of Majorana fermion f^μ, we obtain the commutation relation of Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{15}$$

The vacuum forms a representation of the Clifford algebra (also known as Dirac algebra) $Cl_{1,D-1}$. We see that the constraint $\dot{X}\cdot\psi = 0$ is equivalent to $f_\mu p^\mu = 0$, i.e., $\gamma_\mu p^\mu = \not{p} = 0$, which is nothing but the Dirac equation for massless fermions. □

Problem 4 Derive the fermionic part of the Ramond-Neveu-Schwarz action

$$S_f = \frac{i}{\pi} \int d^2\sigma (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \tag{16}$$

Solution. Recall that the Dirac matrices are of the form

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

We have that

$$\rho^\alpha \partial_\alpha = \begin{pmatrix} 0 & -2\partial_- \\ 2\partial_+ & 0 \end{pmatrix}. \tag{18}$$

The Dirac conjugate of the spinor ψ is $\bar{\psi} = \psi^\dagger i \rho^0$. Thus we have

$$\begin{aligned} \bar{\psi}\rho^\alpha \partial_\alpha \psi = & \begin{pmatrix} \psi_- & \psi_+ \end{pmatrix} i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2\partial_- \\ 2\partial_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \\ = & -2i(\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \end{aligned} \tag{19}$$

Substituting it into the action $-\frac{1}{2\pi}\int d^2\sigma \bar{\psi}\rho^\alpha \partial_\alpha \psi$, we have

$$S_f = \frac{i}{\pi} \int d^2\sigma (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+) \tag{20}$$

□

Problem 5 Prove the following equalities

$$\bar{\psi}_1 \psi_2 = \bar{\psi}_2 \psi_1, \tag{21}$$

$$\bar{\psi}_1 \rho^\alpha \psi_2 = -\bar{\psi}_2 \rho^\alpha \psi_1, \tag{22}$$

$$\bar{\psi}_1 \rho^\alpha \rho^\beta \psi_2 = \bar{\psi}_2 \rho^\beta \rho^\alpha \psi_1. \tag{23}$$

Solution. These are just some straightforward calculation.

$$\bar{\psi}_1 \psi_2 = \psi_1^\dagger i \rho^0 \psi_2 = i \psi_{1A} \rho^0_{AB} \psi_{2B} = -i \rho^0_{BA} \psi_{1A} \psi_{2B} = i \rho^0_{BA} \psi_{2B} \psi_{1A} = \bar{\psi}_2 \psi_1. \tag{24}$$

Note that here we have used the antisymmetric property of ρ^0 and the the anti-commutation relation of Grassmann numbers.

For the second one, notice that $\rho'^\alpha = i\rho^0 \rho^\alpha$ is symmetric for $\alpha = 0, 1$. We have

$$\bar{\psi}_1 \rho^\alpha \psi_2 = \psi_{1A} \rho'^\alpha_{AB} \psi_{2B} = \rho'^\alpha_{BA}(-\psi_{2B}\psi_{1A}) = -\bar{\psi}_2 \rho^\alpha \psi_1. \tag{25}$$

For the last one, recall the anti-commuting relation of Grassmann numbers. To prove the equality, we only need to shown that $(i\rho^0\rho^\alpha\rho^\beta)^T = -i(\rho^0\rho^\beta\rho^\alpha)$. Since

$$\rho^0\rho^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \rho^0\rho^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \rho^1\rho^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \rho^1\rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

Substituting these expression into both sides of $(i\rho^0\rho^\alpha\rho^\beta)^T = -i(\rho^0\rho^\beta\rho^\alpha)$, it is easily checked. Now we have

$$\psi_{1A}(i\rho^0\rho^\alpha\rho^\beta)_{AB}\psi_{2B} = -(i\rho^0\rho^\alpha\rho^\beta)_{AB}\psi_{2B}\psi_{1A} = -i(\rho^0\rho^\beta\rho^\alpha)_{BA}(-\psi_{2B}\psi_{1A}) = \bar{\psi}_2\rho^\beta\rho^\alpha\psi_1 \tag{27}$$

Problem 6 Prove the following two-dimensional Fierz transformation

$$\theta_A \bar{\theta}_B = -\frac{1}{2}\delta_{AB}\bar{\theta}_C\theta_C \tag{28}$$

Solution.
To prove this, let us first introduce two-dimensional anti-symmetric tensor

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_{CD} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{29}$$

We will use the subscripts and superscripts notations for clarity, by definition, we have $\bar{\theta}^B = -i\epsilon^{BE}\theta_E$, the equality what we want to prove now becomes

$$\theta_A \bar{\theta}^B = -\frac{1}{2}\delta_A^B \bar{\theta}^C \theta_C. \tag{30}$$

Recall that for rank-2 anti-symmetric tensors we have

$$\epsilon_{AB}\epsilon^{CD} = (\delta_A^C \delta_B^D - \delta_A^D \delta_B^C), \tag{31}$$

which together with $\epsilon^{EA}\epsilon_{AB} = \delta_B^E$ imply that

$$\delta_B^E \epsilon^{CD} = \epsilon^{EC}\delta_B^D - \epsilon^{ED}\delta_B^C. \tag{32}$$

Now consider the RHS of Eq. (30), and using Eq. (32), we have

$$\begin{aligned}\frac{1}{2}\delta_A^B(-i)\epsilon^{CD}\theta_D\theta_C =& -i\frac{1}{2}(\epsilon^{BC}\delta_A^D - \epsilon^{BD}\delta_A^C)\theta_D\theta_C \\ =& \frac{1}{2}\theta_A\bar{\theta}^B + \frac{1}{2}\theta_A\bar{\theta}^B \\ =& \theta_A\bar{\theta}^B\end{aligned} \tag{33}$$

Note that we have used $\theta_C\theta_D = -\theta_D\theta_C$ in the second step above.

Problem 7 Prove the following equality

$$\{D_A, Q_B\} = 0 \tag{34}$$

Solution.
Recall that

$$D_A = \frac{\partial}{\partial\bar{\theta}^A} + \left(\rho^\alpha\theta\right)_A \partial_\alpha \tag{35}$$

$$Q_B = \frac{\partial}{\partial\bar{\theta}^B} - \left(\rho^\alpha\theta\right)_B \partial_\alpha \tag{36}$$

We have

$$\begin{aligned}\{D_A, Q_B\} =& \{\frac{\partial}{\partial\bar{\theta}^A}, \frac{\partial}{\partial\bar{\theta}^B}\} - \{\frac{\partial}{\partial\bar{\theta}^A}, \left(\rho^\alpha\theta\right)_B \partial_\alpha\} + \{\left(\rho^\alpha\theta\right)_A \partial_\alpha, \frac{\partial}{\partial\bar{\theta}^B}\} - \{\left(\rho^\alpha\theta\right)_A \partial_\alpha, \left(\rho^\alpha\theta\right)_B \partial_\alpha\} \\ =& 0 - \{\frac{\partial}{\partial\bar{\theta}^A}, \left(\rho^\alpha\theta\right)_B \partial_\alpha\} + \{\left(\rho^\alpha\theta\right)_A \partial_\alpha, \frac{\partial}{\partial\bar{\theta}^B}\} - 0\end{aligned} \tag{37}$$

here, the first and the last term vanishes just because that $\{\partial_A, \partial_B\} = 0$ and $\{\theta_A, \theta_B\} = 0$ for Grassmann numbers. Now let's consider the remaining two terms. Recall that $\partial_{\bar{\theta}^A}\bar{\theta}_B = \delta_{AB}$ and $\theta_C = i\rho^0_{CD}\bar{\theta}_D$, we have

$$\begin{aligned}\{\frac{\partial}{\partial\bar{\theta}^A}, \left(\rho^\alpha\theta\right)_B \partial_\alpha\} =& \{\frac{\partial}{\partial\bar{\theta}^A}, \rho^\alpha_{BC}\theta_C\partial_\alpha\} = \{\frac{\partial}{\partial\bar{\theta}^A}, \rho^\alpha_{BC} i\rho^0_{CD}\bar{\theta}_D\partial_\alpha\} \\ =& \rho^\alpha_{BC} i\rho^0_{CD}\{\frac{\partial}{\partial\bar{\theta}^A}, \bar{\theta}_D\}\partial_\alpha\end{aligned} \tag{38}$$

Problem 7 continued on next page...

$$\begin{aligned}\{(\rho^\alpha\theta)_A\,\partial_\alpha,\frac{\partial}{\partial\bar{\theta}^B}\} &= \{\rho^\alpha_{AE}i\rho^0_{EF}\bar{\theta}_F\partial_\alpha,\frac{\partial}{\partial\bar{\theta}^B}\}\\ &= \rho^\alpha_{AE}i\rho^0_{EF}\{\bar{\theta}_F,\frac{\partial}{\partial\bar{\theta}^B}\}\partial_\alpha\end{aligned} \tag{39}$$

With these results and the anti-commutator for $\partial_{\bar{\theta}^A},\bar{\theta}_B$, we arrive at the result that $\{D_A,Q_B\}=0$. $\square$