Problem 1 Prove that

$$\bar{D}Y^\mu = \bar{\psi}^\mu + \bar{\theta}B^\mu - \bar{\theta}\rho^\alpha\partial_\alpha X^\mu + \frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha \tag{1}$$

Solution.

Recall that $\bar{D}_A = -\frac{\partial}{\partial\bar{\theta}^A} - (\bar{\theta}\rho^\alpha)^A\partial_\alpha$ and $Y^\mu(\sigma^\alpha,\theta_A) = X^\mu(\sigma^\alpha) + \bar{\theta}\psi^\mu(\sigma^\alpha) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma^\alpha)$, we have

$$\begin{aligned}\bar{D}^A Y^\mu =&(-\frac{\partial}{\partial\theta_A} - (\bar{\theta}\rho^\alpha)^A\partial_\alpha)Y^\mu\\ =&(-\frac{\partial}{\partial\theta_A}Y^\mu) + (-(\bar{\theta}\rho^\alpha)^A\partial_\alpha Y^\mu)\\ =&(0+\bar{\psi}^\mu_A + \bar{\theta}^A B^\mu) + (-\bar{\theta}\rho^\alpha\partial_\alpha X^\mu - (\bar{\theta}\rho^\alpha)^A\partial_\alpha\bar{\theta}\psi^\mu(\sigma^\alpha)+0)\end{aligned} \tag{2}$$

We have used the fact that, for two-dimensional Grassman algebra, product of three or more Grassman number must vanish. Let us now consider the term $(\bar{\theta}\rho^\alpha)^A\partial_\alpha\bar{\theta}\psi^\mu(\sigma^\alpha)$.

$$\begin{aligned}(\bar{\theta}\rho^\alpha)^A\partial_\alpha\bar{\theta}\psi^\mu(\sigma^\alpha) =&\theta_C(i\rho_0)_{CD}\rho^\alpha_{DA}\bar{\theta}_E\partial_\alpha\psi^\mu_E\\ =&(i\rho_0)_{CD}\rho^\alpha_{DA}(-\frac{1}{2}\delta_{CE}\bar{\theta}_F\theta_F)\partial_\alpha\psi^\mu_E\\ =&-\frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha\end{aligned} \tag{3}$$

Substitute this into the original equation, we obtain

$$\bar{D}Y^\mu = \bar{\psi}^\mu + \bar{\theta}B^\mu - \bar{\theta}\rho^\alpha\partial_\alpha X^\mu + \frac{1}{2}\bar{\theta}\theta\partial_\alpha\bar{\psi}^\mu\rho^\alpha \tag{4}$$

□

Problem 2 Derive the mass formulas for states in the R and NS sector of the RNS open superstring.

Solution.

Let us first consider R sector, the zero mode Virasoro constraint for $a_R = 0$ is

$$L_0 - 0 = 0,$$

where the Virasoro operator is given by

$$L_0 = \frac{1}{2}\alpha_0^2 + N, \tag{5}$$

where number operator is

$$N = \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{n>0} n d^i_{-n}d^i_n. \tag{6}$$

Therefore the mass formula is

$$\alpha' M^2 = \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{n>0} n d^i_{-n}d^i_n. \tag{7}$$

Let us now consider the NS sector, zero-mode Virasoro constraint with $a_{NS} = \frac{1}{2}$ is

$$L_0 - \frac{1}{2} = 0 \tag{8}$$

where Virasoro operator L_0 is given by

$$L_0 = \frac{1}{2}\alpha_0^2 + N \tag{9}$$

with number operator

$$N = \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{r>0} r b^i_{-r}b^i_r \tag{10}$$

Therefore we get mass formula

$$\alpha' M^2 = N - \frac{1}{2} \tag{11}$$

□

Problem 3 Prove that for R sector

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \tfrac{D}{8}m^3\delta_{m,-n} \\
&[L_m, F_n] = \left(\tfrac{m}{2} - n\right) F_{m+n} \\
&\{F_m, F_n\} = 2L_{m+n} + \tfrac{D}{2}m^2\delta_{m,-n}
\end{aligned} \tag{12}$$

Prove that for NS sector

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \tfrac{D}{8}m\left(m^2-1\right)\delta_{m,-n} \\
&[L_m, G_r] = \left(\tfrac{m}{2} - r\right) G_{m+r} \\
&\{G_r, G_s\} = 2L_{r+s} + \tfrac{D}{2}\left(r^2 - \tfrac{1}{4}\right)\delta_{r,-s}
\end{aligned} \tag{13}$$

Solution. For R sector, recall that $L_m = L_m^f + L_m^b$, the bosonic part obey the usual Virasoro commutation relations,

$$[L_m^b, L_m^b] = (m-n)L_{m+n} = \frac{D}{12}(m^3 - m)\delta_{m+n,0}. \tag{14}$$

Let us calculate the fermionic part.

$$L_m^f = \frac{1}{2}\left(\sum_{k\geq -m/2}(k+\frac{m}{2})d_{-k}d_{m+k} + \sum_{k<-m/2}(k+\frac{m}{2})d_{m+k}d_{-k}\right) \tag{15}$$

From which we have

$$\begin{aligned}
[L_m^f, L_n^f] =& \frac{1}{4}\left(\sum_{k\geq -m/2}(k+\frac{m}{2})d_{-k}d_{m+k} + \sum_{k<-m/2}(k+\frac{m}{2})d_{m+k}d_{-k}\right)\left(\sum_{l\geq -n/2}(l+\frac{n}{2})d_{-l}d_{n+l} + \sum_{l<-n/2}(l+\frac{n}{2})d_{n+l}d_{-l}\right) \\
&- \frac{1}{4}\left(\sum_{l\geq -n/2}(l+\frac{n}{2})d_{-l}d_{n+l} + \sum_{l<-n/2}(l+\frac{n}{2})d_{n+l}d_{-l}\right)\left(\sum_{k\geq -m/2}(k+\frac{m}{2})d_{-k}d_{m+k} + \sum_{k<-m/2}(k+\frac{m}{2})d_{m+k}d_{-k}\right)
\end{aligned} \tag{16}$$

Using the commutation relation $\{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}$, we obtain that

$$[L_m^f, L_n^f] = (m-n)L_{m+n}^f + \frac{D}{24}m^3 + \frac{D}{12}m. \tag{17}$$

Since L_m^f and L_n^b commutes (they origin from independent freedoms of the theory), thus we have

$$[L_m, L_n] = [L_m^b, L_m^b] + [L_m^f, L_n^f] = (m-n)L_{m+n} + \frac{D}{8}m\left(m^2-1\right)\delta_{m,-n}. \tag{18}$$

Recall that

$$F_m = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma} J_+ = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n} \tag{19}$$

We have

$$\begin{aligned}
[L_m, F_n] &= \sum_{k\in\mathbb{Z}}[L_m, \alpha_{-k}d_{n+k}] \\
&= \sum_{k\in\mathbb{Z}}(\alpha_{-k}[L_m, d_{n+k}] + [L_m, \alpha_{-k}]d_{n+k}
\end{aligned} \tag{20}$$

Since we have

$$[L_m, \alpha_{-k}^\mu] = [L_m^b, \alpha_{-k}^\mu] = -k\alpha_{m-k}^\mu \tag{21}$$

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$$[L_m, d^\mu_{n+k}] = [L^f_m, d^\mu_{n+k}] = (\frac{3m}{2} + n + k) d^\mu_{m+n+k} \tag{22}$$

Substituting them into the original equation, we obtain that

$$[L_m, F_n] = \left(\frac{m}{2} - n\right) F_{m+n} \tag{23}$$

For the last one,

$$\begin{aligned}
\{F_m, F_n\} &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \{\alpha_{-k} d_{m+k}, \alpha_{-l} d_{n+l}\} \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \alpha^\mu_{-k} d_{m+k,\mu} \alpha^\nu_{-l} d_{n+l,\nu} + \alpha^\nu_{-l} d_{n+l,\nu} \alpha^\mu_{-k} d_{m+k,\mu} \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \alpha^\mu_{-k} \alpha^\nu_{-l} (\eta_{\mu\nu}\delta_{m+k+n+l,0} - d_{n+l,\nu} d_{m+k,\mu}) + \alpha^\nu_{-l}\alpha^\mu_{-k} d_{n+l,\nu} d_{m+k,\mu}) \\
&= 2L_{m+n} + \frac{D}{2} m^2 \delta_{m,-n}
\end{aligned} \tag{24}$$

For the NS sector, recall that

$$L^{\mathrm{f}}_m = \frac{1}{2}\sum_{r\in\mathbb{Z}+1/2} \left(r + \frac{m}{2}\right) : b_{-r}\cdot b_{m+r} : \quad m \in \mathbb{Z}$$

For the first one, we only need to calculate the bosonic part.

$$L^f_m = \frac{1}{2}\left(\sum_{r\geq -m/2} (r + \frac{m}{2}) b_{-r} b_{m+r} + \sum_{r<-m/2} (r + \frac{m}{2}) b_{m+r} b_{-r}\right) \tag{25}$$

From which we have

$$\begin{aligned}
[L^f_m, L^f_n] =& \frac{1}{4}\left(\sum_{r\geq -m/2} (r + \frac{m}{2}) b_{-r} b_{m+r} + \sum_{r<-m/2} (r + \frac{m}{2}) b_{m+r} b_{-r}\right)\left(\sum_{l\geq -n/2} (l + \frac{n}{2}) b_{-l} b_{n+l} + \sum_{l<-n/2} (l + \frac{n}{2}) b_{n+l} b_{-l}\right) \\
&- \frac{1}{4}\left(\sum_{l\geq -n/2} (l + \frac{n}{2}) b_{-l} b_{n+l} + \sum_{l<-n/2} (l + \frac{n}{2}) b_{n+l} b_{-l}\right)\left(\sum_{r\geq -m/2} (r + \frac{m}{2}) b_{-r} b_{m+r} + \sum_{r<-m/2} (r + \frac{m}{2}) b_{m+r} b_{-r}\right)
\end{aligned} \tag{26}$$

Note that $r, l \in \mathbb{Z} + 1/2$. Using the commutation relation $\{b^\mu_r, b^\nu_l\} = \eta^{\mu\nu}\delta_{r+l,0}$, we obtain that

$$[L^f_m, L^f_n] = (m-n) L^f_{m+n} + \frac{D}{24}(m^3 - m)\delta_{m+n,0}. \tag{27}$$

Since L^f_m and L^b_n commutes (they origin from independent freedoms of the theory), thus we have

$$[L_m, L_n] = [L^b_m, L^b_m] + [L^f_m, L^f_n] = (m-n) L_{m+n} + \frac{D}{8} m \left(m^2 - 1\right) \delta_{m,-n}. \tag{28}$$

For the second one, since

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \quad r \in \mathbb{Z} + \frac{1}{2}$$

Thus we have

$$[L_m, G_r] = \sum_{k\in\mathbb{Z}} [L_m, \alpha_{-k} b_{r+k}] \tag{29}$$

$$= \sum_{k\in\mathbb{Z}} \alpha_{-k}[L_m, b_{r+k}] + [L_m, \alpha_{-k}] b_{r+k} \tag{30}$$

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Since we have

$$[L_m, \alpha^\mu_{-k}] = [L^b_m, \alpha^\mu_{-k}] = -k\alpha^\mu_{m-k} \tag{32}$$

$$[L_m, b^\mu_{r+k}] = [L^f_m, b^\mu_{r+k}] = (\frac{3m}{2} + r + k)b^\mu_{m+r+k} \tag{33}$$

Substituting them into the original equation, we obtain

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r} \tag{34}$$

For the last one,

$$\begin{aligned}
\{G_r, G_s\} &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\{\alpha_{-k}b_{r+k}, \alpha_{-l}b_{s+l}\} \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \alpha^\mu_{-k}b_{r+k,\mu}\alpha^\nu_{-l}b_{s+l,\nu} + \alpha^\nu_{-l}d_{s+l,\nu}\alpha^\mu_{-k}b_{r+k,\mu} \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \alpha^\mu_{-k}\alpha^\nu_{-l}(\eta_{\mu\nu}\delta_{r+k+s+l,0} - d_{s+l,\nu}d_{r+k,\mu}) + \alpha^\nu_{-l}\alpha^\mu_{-k}d_{s+l,\nu}d_{r+k,\mu}) \\
&= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}
\end{aligned} \tag{35}$$

□

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