Problem 1 Verify the constants $a_R = 0$, $a_{NS} = 1/2$ for the critical RNS superstring by using zeta-function regularization to compute the world-sheet fermion zero-point energies, as suggested in Section 4.6.

Solution.

Let us first consider the R sector. In the light-cone quantization

$$\begin{aligned} L_0 =& \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{i=1}^{D-2} \alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{i=1}^{D-2} n d^i_{-n} d^i_n \\ =& \frac{1}{2}\alpha_0^2 + \sum_{n>0}\sum_{i=1}^{D-2}\left(\alpha^i_{-n}\alpha^i_n + n d^i_{-n}d^i_n\right) + \frac{1}{2}\sum_{n>0}\sum_{i=1}^{D-2}\left(\left[\alpha^i_n, \alpha^i_{-n}\right] - n\left\{d^i_n, d^i_{-n}\right\}\right) \\ =& \sum_{n>0}\sum_{i=1}^{D-2}\left(\alpha^i_{-n}\alpha^i_n + n d^i_{-n}d^i_n\right) + \frac{1}{2}\alpha_0^2 \end{aligned} \tag{1}$$

Using the condition that $L_0 - a_R = 0$, we have

$$\alpha' M^2 = \sum_{n>0}\sum_{i=1}^{D-2}\left(\alpha^i_{-n}\alpha^i_n + n d^i_{-n}d^i_n\right) - a_R \tag{2}$$

The zero-point energy is 0, from which we have $a_R = 0$.

Similarly, for NS sector, we have

$$L_0 = \frac{1}{2}\sum_{n}\sum_{i=1}^{D-2}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{r}\sum_{i=1}^{D-2} r b^i_{-r} b^i_r \tag{3}$$

$$= \sum_{n>0}\sum_{i=1}^{D-2}\alpha^i_{-n}\alpha^i_n + \sum_{r>0}\sum_{i=1}^{D-2} r b^i_{-r}b^i_r + \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n>0}\sum_{i=1}^{D-2}\left[\alpha^i_n, \alpha^i_{-n}\right] - \frac{1}{2}\sum_{r>0}\sum_{i=1}^{D-2} r\left\{b^i_r, b^i_{-r}\right\} \tag{4}$$

which implies

$$a_{NS} = -\frac{D-2}{2}\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right) = -\frac{D-2}{2}\left(\sum_{n=1}^{\infty} n - \sum_{m=0}^{\infty}(m+1/2)\right) = \frac{D-2}{4}\sum_{m=0}^{\infty} 1$$

Consider zeta-function regularization for $\left(\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r\right)$, we known that $\zeta(-1) = -1/12$, we can rearrange the it as

$$\sum_{k=1}^{\infty}(2k) + \sum_{k=0}^{\infty}(2k+1) = 2\zeta(-1) + 2\zeta(-1) + \sum_{k=1}^{\infty} 1 = \zeta(-1) = -1/12 \tag{5}$$

this implies that $\sum_{k=0}^{\infty} 1 = -3\zeta(-1) = 1/4$. Therefore, we have $a_{NS} = \frac{D-2}{16}$. Substituting the critical dimension $D = 10$, we have $a_{NS} = \frac{1}{2}$, □

Problem 2 Consider the RNS string in ten-dimensional Minkowski space-time. Show that after the GSO projection the NS and R sectors have the same number of physical degrees of freedom at the second massive level. Determine the explicit form of the states in the light-cone gauge. In other words, repeat the analysis of Exercise 4.10 for the next level.

Solution. For the R sector we have the following second massive level states

state	number of degree of freedom
$d^i_{-2}\lvert\psi_0\rangle$	8×8
$d^i_{-1}d^j_{-1}\lvert\psi_0\rangle$	28×8
$\alpha^i_{-2}\lvert\psi_0\rangle$	8×8
$\alpha^i_{-1}\alpha^j_{-1}\lvert\psi_0\rangle$	36×8
$d^i_{-1}\alpha^j_{-1}\lvert\psi_0\rangle$	64×8

Problem 2 continued on next page...

The total number of degree of freedom is 1152.

For the NS sector we have the following second massive level states

state	number of degree of freedom
$b^i_{-5/2}\|0\rangle$	8
$b^i_{-3/2}\alpha^j_{-1}\|0\rangle$	64
$b^i_{-1/2}\alpha^j_{-2}\|0\rangle$	64
$b^i_{-1/2}\alpha^j_{-1}\alpha^k_{-1}\|0\rangle$	288
$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}\alpha^l_{-1}\|0\rangle$	448
$b^i_{-1/2}b^j_{-1/2}b^k_{-1/2}b^l_{-1/2}b^m_{-1/2}\|0\rangle$	56
$b^i_{-3/2}b^j_{-1/2}b^k_{-1/2}\|0\rangle$	224

Thus there in total 1152 degree of freedom which is the same as the R sector case. □

Problem 3 Given a pair of two-dimensional Majorana spinors ψ and χ, prove that

$$\psi_A\bar{\chi}_B = -\frac{1}{2}\left(\bar{\chi}\psi\delta_{AB} + \bar{\chi}\rho_\alpha\psi\rho^\alpha_{AB} + \bar{\chi}\rho_3\psi\left(\rho_3\right)_{AB}\right)$$

where $\rho_3 = \rho_0\rho_1$

Solution. This can be checked by direct calculation.

$$\begin{aligned} RHS = &-\frac{1}{2}\left(\chi_C i\rho^0_{CD}\psi_D\delta_{AB} + \chi_C i\rho^0_{CD}(\rho_\alpha)_{DE}\psi_E\rho^\alpha_{AB} + \chi_C i\rho^0_{CD}(\rho_0\rho_1)_{DE}\psi_E(\rho_0\rho_1)_{AB}\right) \\ &= -i\frac{1}{2}\left(\left(\chi_2\psi_1 - \chi_1\psi_2\right)\delta_{AB} + \left(\chi_1\psi_1 + \chi_2\psi_2\right)\rho^0_{AB} + \left(\chi_2\psi_2 - \chi_1\psi_1\right)\rho^1_{AB} + \left(\chi_2\psi_1 + \chi_1\psi_2\right)(\rho_0\rho_1)_{AB}\right) \end{aligned} \tag{6}$$

Meanwhile, the left-hand side is of the form

$$LHS = \psi_A\chi_C i\rho^0_{CB} \tag{7}$$

by comparing two equations for all possible $A, B = 0, 1$, the required equation is obviously true. □

Problem 4 Derive the NS-sector Lorentz transformation generators in the light-cone gauge.

Solution. The Lorentz generator is the Noether charge corresponds to the Lorentz transformation. Note that the bosonic part has been done in bosonic string chapter, we here consider the fermionic part $S_f \sim \int d^2\sigma\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu$. Recall that for a spinor field $\psi(x)$ on spacetime manifold, under Lorentz transformation Λ, it transforms as

$$\psi(x) \to \Lambda_{1/2}\psi(\Lambda^{-1}x). \tag{8}$$

Here the filed is defined on world-sheet manifold, thus under Lorentz transformation, σ^α remains unchanged. Consider the infinitesimal Lorentz transformation characterized by the parameter $\omega_{\mu\nu}$, the corresponding Noether current is

$$J^\alpha = \frac{\partial\mathcal{L}}{\partial\partial_\alpha\psi^\rho}\delta\psi^\rho. \tag{9}$$

From this we obtain that

$$J^\alpha_{\mu\nu} = \frac{1}{2}\left(\bar{\psi}_\nu\rho^\alpha\psi_\mu - \bar{\psi}_\mu\rho^\alpha\psi_\nu\right). \tag{10}$$

The Noether charge is thus

$$J_{\mu\nu} = \int_0^\pi d\sigma J^0_{\mu\nu} = \frac{1}{2}\int_0^\pi d\sigma\left(\bar{\psi}_\nu\rho^0\psi_\mu - \bar{\psi}_\mu\rho^0\psi_\nu\right) = \frac{i}{2}\int_0^\pi d\sigma\left(\psi^T_\nu\psi_\mu - \psi^T_\mu\psi_\nu\right) \tag{11}$$

Problem 4 continued on next page. . .

Recall the for NS sector we have the mode expansion

$$\psi_-^\mu(\sigma^-) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir\sigma^-}, \tag{12}$$

$$\psi_+^\mu(\sigma^+) = \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+1/2} b_r^\mu e^{-ir\sigma^+}. \tag{13}$$

Substituting them in the expression of Noether current, we obtain

$$\begin{aligned} J^{\mu\nu} =& \frac{i}{2}\int_0^\pi d\sigma \sum_{r,s}\left(b_r^\nu b_s^\mu - b_r^\mu b_s^\nu\right)\cos((r+s)\sigma)e^{-i(r+s)\tau} \\ =& \frac{i}{2}\sum_r \left(b_r^\nu b_{-r}^\mu - b_r^\mu b_{-r}^\nu\right) \\ =& i\sum_{r>0}\left(b_r^\nu b_{-r}^\mu - b_r^\mu b_{-r}^\nu\right) \end{aligned} \tag{14}$$

Now let's translate the expression into the language of light-cone quantization. Since $b_r^+ = 0$, (we use $I, J = 2, \cdots, 9$ as light-cone indices)

$$J^{+-} = 0, J^{+I} = 0, J^{-I} = 0. \tag{15}$$

and

$$J^{IJ} = i\sum_{r>0}\left(b_r^I b_{-r}^J - b_r^J b_{-r}^I\right). \tag{16}$$

□

Problem 5 Prove that

$$\begin{aligned} \{\mathcal{G}_r, \mathcal{G}_s\} &= ie^{2i(r+s)\sigma^-}\left(\partial_- - (r+s)\theta_+ \frac{\partial}{\partial\bar{\theta}^-}\right) \\ &= 2\mathcal{L}_{r+s} \\ [\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} \\ [\mathcal{L}_n, \mathcal{G}_r] &= \left(\frac{n}{2} - r\right)\mathcal{G}_{n+r} \end{aligned}$$

where $r+s$ is an integer and

$$\begin{aligned} \mathcal{L}_n &= \frac{i}{2}e^{2in\sigma^-}\left(\partial_- - n\theta_+ \frac{\partial}{\partial\bar{\theta}^-}\right) \\ \mathcal{G}_r &= -\frac{i}{2}e^{2ir\sigma^-}\left(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-\right) \end{aligned}$$

Solution.
For the first one, let's first calcuate

$$\begin{aligned} \mathcal{G}_r\mathcal{G}_s =& \frac{i}{2}e^{2ir\sigma^-}\left(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-\right)\frac{i}{2}e^{2is\sigma^-}\left(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-\right) \\ =& ie^{2i(r+s)\sigma^-}(\frac{1}{2}\partial_- - s\theta_+\frac{\partial}{\partial\bar{\theta}^-}) \end{aligned} \tag{17}$$

exchange indices r, s, we have $\mathcal{G}_s\mathcal{G}_r$, from them, it is obvious that

$$\{\mathcal{G}_r, \mathcal{G}_s\} = ie^{2i(r+s)\sigma^-}\left(\partial_- - (r+s)\theta_+\frac{\partial}{\partial\bar{\theta}^-}\right) \tag{18}$$

$$= 2\mathcal{L}_{r+s} \tag{19}$$

$$\tag{20}$$

Problem 5 continued on next page. . .

Now, let us consider the second one. We can also first calculate

$$\begin{aligned}\mathcal{L}_n\mathcal{L}_m =&\frac{i}{2}e^{2in\sigma^-}\left(\partial_- - n\theta_+\frac{\partial}{\partial\bar{\theta}^-}\right)\frac{i}{2}e^{2im\sigma^-}\left(\partial_- - m\theta_+\frac{\partial}{\partial\bar{\theta}^-}\right)\\ =&-\frac{1}{4}e^{2i(m+n)\sigma^-}\left[2im(\partial_- - m\theta_+\frac{\partial}{\partial\bar{\theta}^-}) + \partial_-^2 - (m+n)\theta_+\frac{\partial}{\partial\bar{\theta}^-}\partial_- + mn\frac{\partial}{\partial\bar{\theta}^-}\theta_+\frac{\partial}{\partial\bar{\theta}^-}\right]\end{aligned} \tag{21}$$

Exchanging the indices m, n, we obtain the expression of $\mathcal{L}_m\mathcal{L}_n$, combining these results, we have

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}. \tag{22}$$

For the last one,

$$\begin{aligned}\mathcal{L}_n\mathcal{G}_r =&\frac{1}{4}e^{2in\sigma^-}\left(\partial_- - n\theta_+\frac{\partial}{\partial\bar{\theta}^-}\right)e^{2ir\sigma^-}\left(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-\right)\\ =&\frac{1}{4}e^{2i(n+r)\sigma^-}\left[2ir(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-) + \partial_-(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-) - n\theta_+\frac{\partial}{\partial\bar{\theta}^-}(\frac{\partial}{\partial\bar{\theta}^-} + 2\theta_+\partial_-)\right]\end{aligned} \tag{23}$$

similarly, we can obtain the expression of $\mathcal{G}_r\mathcal{L}_n$, from which we have

$$[\mathcal{L}_n, \mathcal{G}_r] = \left(\frac{n}{2} - r\right)\mathcal{G}_{n+r} \tag{24}$$