Problem 1 Prove the following OPEs

$$
\begin{aligned}
T_B(z)T_B(w) &= \tfrac{3D/4}{(z-w)^4} + \tfrac{2}{(z-w)^2}T_B(w) + \tfrac{1}{z-w}\partial_w T_B(w) + \cdots \\
T_B(z)T_F(w) &= \tfrac{3/2}{(z-w)^2}T_F(w) + \tfrac{1}{z-w}\partial_w T_F(w) + \cdots \\
T_F(z)T_B(w) &= \tfrac{3/2}{(z-w)^2}T_F(w) + \tfrac{1/2}{z-w}\partial_w T_F(w) + \cdots \\
T_F(z)T_F(w) &= \tfrac{D}{(z-w)^3} + \tfrac{2}{z-w}T_B(w) + \cdots
\end{aligned}
\tag{1}
$$

Solution. Recall that

$$
\begin{aligned}
&T_F(z) = 2i\psi(z)\cdot\partial X(z) \\
&T_B(z) = -2 : \partial X(z)\cdot\partial X(z) : -\tfrac{1}{2} : \psi(z)\cdot\partial\psi(z) :
\end{aligned}
\tag{2}
$$

and the OPE

$$
X^\mu(z)X^\nu(w) \sim -\frac{1}{4}\eta^{\mu\nu}\ln(z-w), \quad \psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}. \tag{3}
$$

From which we have

$$
(\partial X^\mu(z))X^\nu(w) \sim -\frac{1}{4}\frac{\eta^{\mu\nu}}{z-w}, \tag{4}
$$

$$
X^\mu(z)(\partial X^\mu(w)) \sim \frac{1}{4}\frac{\eta^{\mu\nu}}{z-w}, \tag{5}
$$

$$
(\partial X^\mu(z))(\partial X^\nu(w)) \sim -\frac{1}{4}\frac{\eta^{\mu\nu}}{(z-w)^2}, \tag{6}
$$

and

$$
(\partial\psi^\mu(z))\psi^\nu(w) \sim -\frac{\eta^{\mu\nu}}{(z-w)^2}, \tag{7}
$$

$$
\psi^\mu(z)(\partial\psi^\mu(w)) \sim \frac{\eta^{\mu\nu}}{(z-w)^2}, \tag{8}
$$

$$
(\partial\psi^\mu(z))(\partial\psi^\nu(w)) \sim -2\frac{\eta^{\mu\nu}}{(z-w)^3} \tag{9}
$$

Define

$$
T_{B,1} = -2 : \partial X(z)\cdot\partial X(z) :, \quad T_{B,2} = -\frac{1}{2} : \psi(z)\cdot\partial\psi(z) : \tag{10}
$$

then $T_B = T_{B,1} + T_{B,2}$. Recall the we have calculated the OPE for $T_{B,1}$ in bosonic string theory

$$
T_{B,1}(z)T_{B,1}(w) \sim \frac{D/2}{(z-w)^4} + \frac{2}{(z-w)^2}T_{B,1}(w) + \frac{1}{z-w}\partial T_{B,1}(w) \tag{11}
$$

For the $T_{B,2}$ part, we have

$$
\begin{aligned}
T_{B,1}(z)T_{B,1}(w) =& \frac{1}{4} : \psi(z)\cdot\partial\psi(z) :: \psi(w)\cdot\partial\psi(w) : \\
\sim& \frac{1}{4}[\psi^\mu(z)\partial\psi^\nu\eta_{\mu\nu}\psi^\rho(w)\partial\psi^\sigma(w)\eta_{\rho\sigma} + \psi^\mu(z)\partial\psi^\nu\eta_{\mu\nu}\psi^\rho(w)\partial\psi^\sigma(w)\eta_{\rho\sigma} \\
&+ \psi^\mu(z)\partial\psi^\nu\eta_{\mu\nu}\psi^\rho(w)\partial\psi^\sigma(w)\eta_{\rho\sigma} + \psi^\mu(z)\partial\psi^\nu\eta_{\mu\nu}\psi^\rho(w)\partial\psi^\sigma(w)\eta_{\rho\sigma} \\
&+ \psi^\mu(z)\partial\psi^\nu\eta_{\mu\nu}\psi^\rho(w)\partial\psi^\sigma(w)\eta_{\rho\sigma} + \psi^\mu(z)\partial\psi^\nu(z)\eta_{\mu\nu}\psi^\rho(w)\partial\psi^\sigma(w)\eta_{\rho\sigma}] \\
\sim& \frac{D/4}{(z-w)^4} + \frac{2}{(z-w)^2}T_{B,2}(w) + \frac{1}{z-w}\partial T_{B,2}(w)
\end{aligned}
\tag{12}
$$

Problem 1 continued on next page...

Here we have used Eqs. (3) to (9) and Taylor expansion for $T(z)$ and $\partial T(z)$ around w and omitted the regular terms. From these two OPEs, we have

$$\begin{aligned}
T_B(z)T_B(w) =&(T_{B,1}+T_{B,2})(T_{B,1}+T_{B,2})\\
=&T_{B,1}T_{B,1}+T_{B,2}T_{B,2}\\
\sim&\frac{3D/4}{(z-w)^4}+\frac{2}{(z-w)^2}T_B(w)+\frac{1}{z-w}\partial_w T_B(w)
\end{aligned}\tag{13}$$

For the OPE of

$$\begin{aligned}
T_F(z)T_F(w) =&-4\psi(z)\cdot\partial X(z)\psi(w)\partial X(w)\\
\sim&-4[\eta_{\mu\nu}\psi^{\mu}(z)\partial X^{\nu}(z)\eta_{\rho\sigma}\psi^{\rho}(w)\partial X^{\sigma}(w)+\eta_{\mu\nu}\psi^{\mu}(z)\partial X^{\nu}(z)\eta_{\rho\sigma}\psi^{\rho}(w)\partial X^{\sigma}(w)\\
&+\eta_{\mu\nu}\psi^{\mu}(z)\partial X^{\nu}(z)\eta_{\rho\sigma}\psi^{\rho}(w)\partial X^{\sigma}(w)\\
=&-4[\eta_{\mu\nu}\eta_{\rho\sigma}\frac{\eta^{\mu\rho}}{(z-w)}:\partial X^{\nu}(z)\partial X^{\sigma}(z):+\eta_{\mu\nu}\eta_{\rho\sigma}(-\frac{\eta^{\nu\sigma}}{4})\frac{1}{(z-w)^2}:\psi^{\mu}(z)\psi^{\rho}(w):-\frac{1}{4}\frac{D}{(z-w)^3}]\\
\sim&\frac{D}{(z-w)^3}+\frac{2}{z-w}T_B(w)
\end{aligned}\tag{14}$$

Notice that in the last step we have used the Taylor expansion of $\partial X(z)$ and $\psi^{\mu}(z)$ around $z=w$ and omitted the regular terms.

For the OPE,

$$\begin{aligned}
T_B(z)T_F(w) =&-4i:\partial X(z)\cdot\partial X(z):\psi(w)\cdot\partial X(w)-i:\psi(z)\cdot\partial\psi(z):\psi(w)\cdot\partial X(w)\\
\sim&-8i\eta_{\mu\nu}\partial X^{\mu}(z)\partial X^{\nu}(z)\psi^{\rho}(w)\partial X^{\sigma}(w)\eta_{\rho\sigma}-i\eta_{\mu\nu}\psi^{\mu}(z)\cdot\partial\psi^{\nu}(z)\psi^{\rho}(w)\cdot\partial X^{\sigma}(w)\eta_{\rho\sigma}\\
&-i\eta_{\mu\nu}\psi^{\mu}(z)\cdot\partial\psi^{\nu}(z)\psi^{\rho}(w)\cdot\partial X^{\sigma}(w)\eta_{\rho\sigma}\\
=&-8i\eta_{\mu\nu}(-\frac{\eta^{\mu\sigma}}{4})\frac{1}{(z-w)^2}\partial X^{\nu}(z)\psi^{\rho}(w)-i\eta_{\mu\nu}\frac{\eta^{\mu\rho}}{z-w}\psi^{\nu}(z)\partial X^{\sigma}(w)\eta_{\rho\sigma}\\
&-i\eta_{\mu\nu}(-\frac{\eta^{\nu\rho}}{(z-w)^2})\psi^{\mu}(z)\partial X^{\sigma}(w)\eta_{\rho\sigma}\\
\sim&\frac{3/2}{(z-w)^2}T_F(w)+\frac{1}{z-w}\partial_w T_F(w)
\end{aligned}\tag{15}$$

Notice that in the last step we have used the Taylor expansion of $\partial X^{\nu}(z)$ and $\psi^{\mu}(z),\psi^{\nu}(z)$ around $z=w$ and omitted the regular terms.

For the last one, with completely the same philosophy, we have

$$\begin{aligned}
T_F(z)T_B(w) =&-4i\psi(z)\cdot\partial X(z):\partial X(w)\cdot\partial X(w):-i\psi(z)\cdot\partial X(z):\psi(w)\cdot\partial\psi(w):\\
\sim&-8i\eta_{\mu\nu}\psi^{\mu}\partial X^{\nu}(Z)\partial X^{\rho}(w)\partial^{\sigma}(w)\eta_{\rho\sigma}-i\eta^{\mu}\psi^{\mu}(z)\partial X^{\nu}(z)\eta_{\rho\sigma}\psi^{\rho}(w)\partial\psi^{\sigma}(w)\\
&-i\eta^{\mu}\psi^{\mu}(z)\partial X^{\nu}(z)\eta_{\rho\sigma}\psi^{\rho}(w)\partial\psi^{\sigma}(w)\\
\sim&\frac{3/2}{(z-w)^2}T_F(w)+\frac{1/2}{z-w}\partial_w T_F(w)
\end{aligned}\tag{16}$$

where as per usual, we have used Eqs (3) to (9) and Taylor expansion trick.

□

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