Problem 1 Prove that, for a pair of Majorana spinors, Θ_1 and Θ_2, the flip symmetry is given by

$$\bar{\Theta}_1\Gamma_{\mu_1\cdots\mu_n}\Theta_2 = (-1)^{n(n+1)/2}\bar{\Theta}_2\Gamma_{\mu_1\cdots\mu_n}\Theta_1$$

as asserted at the end of Exercise 5.2

Solution. Recall that for Majorana representation of Dirac matrices, we have $\Gamma_\mu^\dagger = \Gamma_\mu^T$, and the charge conjugate is given by $C = \Gamma^0$, for which we have $C\Gamma_\mu = -\Gamma_\mu^T C$.

Since $\Gamma_{\mu_1\cdots\mu_n} := \Gamma_{[\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_n]}$, using the same method as in exercise 5.2, we have

$$\begin{aligned} C\Gamma_{\mu_1\cdots\mu_n}C^{-1} =& C\Gamma_{[\mu_1}\Gamma_{\mu_2}\cdots\Gamma_{\mu_n]}C^{-1} && (1)\\ =& C\Gamma_{[\mu_1}C^{-1}C\Gamma_{\mu_2}C^{-1}\cdots C\Gamma_{\mu_n]}C^{-1} && (2)\\ =& (-1)^n\Gamma^T_{[\mu_1}\Gamma^T_{\mu_2}\cdots\Gamma^T_{\mu_n]} && (3)\\ =& (-1)^n\Gamma^T_{\mu_n\cdots\mu_1} && (4) \end{aligned}$$

therefore

$$C\Gamma_{\mu_1\cdots\mu_n} = (-1)^n\Gamma^T_{\mu_n\cdots\mu_1}C \qquad (5)$$

Now, since $C^T = -C$ (recall that $C = \Gamma^0$) and Θ are Grassmannian, we have

$$\begin{aligned} \bar{\Theta}_1\Gamma_{\mu_1\cdots\mu_n}\Theta_2 =& \bar{\Theta}_1\Gamma_{[\mu_1}\cdots\Gamma_{\mu_n]}\Theta_2 && (6)\\ =& \Theta_1^T C\Gamma_{[\mu_1}\cdots\Gamma_{\mu_n]}\Theta_2 && (7)\\ =& -\Theta_2^T\Gamma^T_{[\mu_n}\cdots\Gamma^T_{\mu_1]}C^T\Theta_1 && (8)\\ =& \Theta_2^T\Gamma^T_{[\mu_n}\cdots\Gamma^T_{\mu_1]}C\Theta_1 && (9)\\ =& \Theta_2^T\Gamma^T_{\mu_1\cdots\mu_n}C\Theta_1 && (10)\\ =& (-1)^n\bar{\Theta}_2\Gamma_{\mu_n\cdots\mu_1}\Theta_1 && (11)\\ =& (-1)^{n+\frac{n(n-1)}{2}}\bar{\Theta}_2\Gamma_{\mu_1\cdots\mu_n}\Theta_1 && (12)\\ =& (-1)^{\frac{n(n+1)}{2}}\bar{\Theta}_2\Gamma_{\mu_1\cdots\mu_n}\Theta_1 && (13) \end{aligned}$$

□

Problem 2 Derive the relevant Fierz transformation identities for Majorana-Weyl spinors in ten dimensions and use them to prove that

$$\Gamma^\mu d\Theta d\bar{\Theta}\Gamma_\mu d\Theta = 0$$

Solution. Since we have

$$\Gamma^\mu d\Theta d\bar{\Theta}\Gamma_\mu d\Theta = \Gamma^\mu\Theta_{,\lambda}\bar{\Theta}_{,\rho}\Gamma_\mu\Theta_{,\sigma}d\sigma^\lambda\wedge d\sigma^\rho\wedge d\sigma^\sigma, \qquad (14)$$

where $d\sigma^\lambda\wedge d\sigma^\rho\wedge d\sigma^\sigma$ is anti-symmetric with respect to indices λ,ρ,σ), thus, it is sufficient to prove that

$$\Gamma^\mu\Theta_{[,\lambda}\bar{\Theta}_{,\rho}\Gamma_\mu\Theta_{,\sigma]} = 0 \qquad (15)$$

This is a special form the general

$$\Gamma^\mu\psi_{[1}\bar{\psi}_2\Gamma_\mu\psi_{3]} = 0 \Leftrightarrow \Gamma^0\Gamma^\mu\psi_{[1}\bar{\psi}_2\Gamma_\mu\psi_{3]} = 0 \qquad (16)$$

Here we will prove it by direct calculation, recall that we have $\bar{\psi}_a\Gamma^\mu\psi_b = -\bar{\psi}_b\Gamma^\mu\psi_a$, using this, we have

$$\begin{aligned} \Gamma^0\Gamma^\mu\psi_{[1}\bar{\psi}_2\Gamma_\mu\psi_{3]} \propto& \varepsilon^{abc}\Gamma^0\Gamma_\mu\psi_a\bar{\psi}_b\Gamma^\mu\psi_c && (17)\\ =& 2\Gamma^0\Gamma_\mu\psi_1\bar{\psi}_2\Gamma^\mu\psi_3 + 2\Gamma^0\Gamma_\mu\psi_2\bar{\psi}_3\Gamma^\mu\psi_1 + 2\Gamma^0\Gamma_\mu\psi_3\bar{\psi}_1\Gamma^\mu\psi_2 && (18)\\ =& 2\Big((\Gamma^0\Gamma_\mu)_{mn}(\Gamma^0\Gamma^\mu)_{pq} + (\Gamma^0\Gamma_\mu)_{mp}(\Gamma^0\Gamma^\mu)_{qn} + (\Gamma^0\Gamma_\mu)_{mq}(\Gamma^0\Gamma^\mu)_{np}\Big)\psi_{1n}\psi_{2p}\psi_{3q} && (19) \end{aligned}$$

Problem 2 continued on next page...

Thus we need to show that

$$\Big((\Gamma^0\Gamma_\mu)_{mn}(\Gamma^0\Gamma^\mu)_{pq}+(\Gamma^0\Gamma_\mu)_{mp}(\Gamma^0\Gamma^\mu)_{qn}+(\Gamma^0\Gamma_\mu)_{mq}(\Gamma^0\Gamma^\mu)_{np}\Big)=0 \tag{20}$$

Notice that $\Gamma^0\Gamma^\mu$ and $\Gamma^0\Gamma_\mu$ are symmetric, thus left hand side of the above equation are symmetric with respect to m,n. Let us take inner produce for arbitrary two spinors $\psi_1 p$ and $\psi_2 q$ of the left hand side of the above equation

$$\left(\Gamma^0\Gamma^\mu\right)_{mn}\bar{\psi}_1\Gamma_\mu\psi_2+\left(\Gamma^0\Gamma^\mu\psi_1\right)_m\left(\bar{\psi}_2\Gamma_\mu\right)_n-\left(\Gamma^0\Gamma^\mu\psi_2\right)_m\left(\bar{\psi}_1\Gamma_\mu\right)_n=A_{mn} \tag{21}$$

This can be regarded as a symmetric matrix with index m,n. Recall that the basis for the matrix space is $\left(\Gamma_{\mu_1\ldots\mu_k}\right)_{mn}$, where $k=0,1,\ldots,10$ in ten spacetime dimensions. We only need to show that the matrix A_{mn} have zero coefficients on each basis. First of all, we note that the terms with k even vanish because of the Weyl projections. Also, the identity

$$\Gamma_{\mu_1\ldots\mu_k}=\pm\frac{1}{(10-k)!}\epsilon_{\mu_1\ldots\mu_{10}}\Gamma^{\mu_{k+1}\cdots\mu_{10}}\Gamma_{11} \tag{22}$$

and the fact that Γ_{11} can be dropped for Weyl spinors implies that only terms with $k\le 5$ need to be considered and that the tensor $\Gamma_{\mu_1\ldots\mu_5}$ can be decomposed into a self-dual and an anti-self-dual piece only one of which contributes. Moreover, $\Gamma^0\Gamma_\mu$ and $\Gamma^0\Gamma_{\mu_1\ldots\mu_5}$ are symmetric, whereas $\Gamma^0\Gamma_{\mu_1\mu_2\mu_3}$ is antisymmetric. Since A_{mn} are symmetric, we only need to consider the $k=1$ and $k=5$ terms. For $k=1$ case

$$\begin{aligned}&\operatorname{tr}\left(\Gamma^\mu\Gamma_\rho\right)\bar{\psi}_1\Gamma_\mu\psi_2-\bar{\psi}_2\Gamma_\mu\Gamma_\rho\Gamma^\mu\psi_1+\bar{\psi}_1\Gamma_\mu\Gamma_\rho\Gamma^\mu\psi_2\\&=-16\bar{\psi}_1\Gamma_\rho\psi_2-8\bar{\psi}_2\Gamma_\rho\psi_1+8\bar{\psi}_1\Gamma_\rho\psi_2=0\end{aligned} \tag{23}$$

For $k=5$ case

$$-\bar{\psi}_2\Gamma_\mu\Gamma_{\rho_1\ldots\rho_5}\Gamma^\mu\psi_1+\bar{\psi}_1\Gamma_\mu\Gamma_{\rho_1\ldots\rho_5}\Gamma^\mu\psi_2=2\bar{\psi}_1\Gamma_\mu\Gamma_{\rho_1\ldots\rho_5}\Gamma^\mu\psi_2 \tag{24}$$

However, in D dimensions,

$$\Gamma_\mu\Gamma_{\rho_1\ldots\rho_k}\Gamma^\mu=(-1)^{k+1}(D-2k)\Gamma_{\rho_1\ldots\rho_k}$$

Taking $D=10$ and $k=5$ we see the above term also equal to zero. Thus conclude that $A_{mn}=0$, this completes the proof. □

Problem 3 Verify that the action (5.41) with Ω_2 given by Eq. (5.55) is invariant under supersymmetry transformations.

Solution.

Recall that the action is

$$S_2=\int_M\Omega_2=\int_D\Omega_3$$

where M is world-sheet manifold, which is the boundary of some manifold D. According to Stokes theorem $\Omega_3=d\Omega_2$. We know that $\Omega_2=c\left(\bar{\Theta}^1\Gamma_\mu d\Theta^1-\bar{\Theta}^2\Gamma_\mu d\Theta^2\right)dX^\mu-c\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2$, from which we have

$$\Omega_3/c=d\Omega_2/c=\left(d\bar{\Theta}^1\Gamma_\mu d\Theta^1-d\bar{\Theta}^2\Gamma_\mu d\Theta^2\right)\left(dX^\mu-\bar{\Theta}^A\Gamma^\mu d\Theta^A\right)+ \tag{25}$$

$$+\left(d\bar{\Theta}^1\Gamma_\mu d\Theta^1-d\bar{\Theta}^2\Gamma_\mu d\Theta^2\right)\left(\bar{\Theta}^1\Gamma^\mu d\Theta^1+\bar{\Theta}^2\Gamma^\mu d\Theta^2\right) \tag{26}$$

$$-d\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2-\bar{\Theta}^1\Gamma_\mu d\Theta^1 d\bar{\Theta}^2\Gamma^\mu d\Theta^2 \tag{27}$$

$$=\tilde{\Omega}_3/c-\bar{\Theta}^1\Gamma^\mu d\Theta^1 d\bar{\Theta}^1\Gamma_\mu d\Theta^1+\bar{\Theta}^2\Gamma^\mu d\Theta^2 d\bar{\Theta}^2\Gamma_\mu d\Theta^2 \tag{28}$$

where

$$\tilde{\Omega}_3=c\left(d\bar{\Theta}^1\Gamma_\mu d\Theta^1-d\bar{\Theta}^2\Gamma_\mu d\Theta^2\right)\Pi^\mu \tag{29}$$

Problem 3 continued on next page...

Since $\tilde{\Omega}_3$ is the product os three supersymmetric one form, $d\Theta^A$ and $\Pi^\mu = dX^\mu - \bar{\Theta}^A\Gamma^\mu d\Theta^A$.

Substitution of supersymmetry transformations

$$\delta\Theta^A = \varepsilon^A, \quad \delta X^\mu = \bar{\varepsilon}^A\Gamma^\mu\Theta^A \tag{30}$$

we have $\delta d\Theta^A = 0$ and

$$\delta\Pi^\mu = \delta dX^\mu - \bar{\varepsilon}\Gamma^\mu d\Theta^A = d\bar{\varepsilon}^A\Gamma^\mu\Theta^A - \bar{\varepsilon}\Gamma^\mu d\Theta^A = 0. \tag{31}$$

Thus, it's a supersymmetric 3-form. Let us consider the term

$$-\bar{\Theta}^1\Gamma^\mu d\Theta^1 d\bar{\Theta}^1\Gamma_\mu d\Theta^1 + \bar{\Theta}^2\Gamma^\mu d\Theta^2 d\bar{\Theta}^2\Gamma_\mu d\Theta^2 \tag{32}$$

Recall that Θ^A are Majorana-Weyl spinors, form problem 2, we see that

$$\Gamma^\mu d\Theta^A d\bar{\Theta}^A\Gamma_\mu d\Theta^A = 0 \tag{33}$$

Therefore $\Omega_3 = \tilde{\Omega}_3$, thus, the action is supersymmetric.

Problem 4 Prove the identity

$$\left\{\Gamma_{\mu_1\nu_1}, \Gamma_{\mu_2\nu_2}\right\} = -2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + 2\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} + 2\Gamma_{\mu_1\nu_1\mu_2\nu_2}$$

invoked in Exercise 5.5

Solution.

Fisrtly, if $\mu_1 = \nu_1$ or $\mu_2 = \nu_2$, both sides are equal to zero.

We now assume that $\mu_1 \neq \nu_1$ and $\mu_2 \neq \nu_2$.

When $\mu_1 = \mu_2 = \alpha$, the left hand side becomes

$$(\Gamma_{\alpha\nu_1}\Gamma_{\alpha\nu_2} + \Gamma_{\alpha\nu_2}\Gamma_{\alpha\nu_1}) = -\Gamma_\alpha^2(\Gamma_{\nu_1}\Gamma_{\nu_2} + \Gamma_{\nu_2}\Gamma_{\nu_1}) - \eta_{\alpha\alpha}2\eta_{\nu_1\nu_2} = -2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} \tag{34}$$

When $\mu_1 = \nu_2 = \alpha$, the left hand side becomes

$$-(\Gamma_{\alpha\nu_1}\Gamma_{\alpha\mu_2} + \Gamma_{\alpha\mu_2}\Gamma_{\alpha\nu_1}) = \Gamma_\alpha^2(\Gamma_{\nu_1}\Gamma_{\mu_2} + \Gamma_{\mu_2}\Gamma_{\nu_1}) - \eta_{\alpha\alpha}2\eta_{\nu_1\mu_2} = 2\eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} \tag{35}$$

When $mu_1 \neq \nu_1 \neq \mu_2 \neq \nu_2$, i.e., for indices are different with each other, the left hand side is

$$\frac{1}{4}(\Gamma_{\mu_1}\Gamma_{\nu_1}\Gamma_{\mu_2}\Gamma_{\nu_2} - \Gamma_{\mu_1}\Gamma_{\nu_1}\Gamma_{\nu_2}\Gamma_{\mu_2} - \Gamma_{\nu_1}\Gamma_{\mu_1}\Gamma_{\mu_2}\Gamma_{\nu_2} + \Gamma_{\nu_1}\Gamma_{\mu_1}\Gamma_{\nu_2}\Gamma_{\mu_2}) + (\text{index } 1 \leftrightarrow 2) \tag{36}$$

For each term in the summation, we have(take the first term as an example)

$$\Gamma_{\mu_1}\Gamma_{\nu_1}\Gamma_{\mu_2}\Gamma_{\nu_2} = \frac{1}{3}(\Gamma_{\mu_1}\Gamma_{\nu_1}\Gamma_{\mu_2}\Gamma_{\nu_2} - \Gamma_{\mu_1}\Gamma_{\mu_2}\Gamma_{\nu_1}\Gamma_{\nu_2} - \Gamma_{\nu_2}\Gamma_{\nu_1}\Gamma_{\mu_2}\Gamma_{\mu_1}) \tag{37}$$

Each term in Eq. (36) can split into three terms in the similar way. The result is nothing but the $2\Gamma_{\mu_1\nu_1\mu_2\nu_2}$ Thus, we complete the proof.

Problem 5 Verify that the action (5.62) is supersymmetric.

Solution. The action can be written as the geometric form as

$$S_2 = \int_M \Omega_2 = \frac{1}{\pi}\int_M \left(\left(\bar{\Theta}^1\Gamma_\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\right) dX^\mu - \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\Theta}^2\Gamma^\mu d\Theta^2\right), \tag{38}$$

where we have assumes wedge products everywhere where differentials (1-forms) are multiplied.

Substitution of supersymmetry transformations

$$\delta\Theta^A = \varepsilon^A, \quad \delta X^\mu = \bar{\varepsilon}^A\Gamma^\mu\Theta^A$$

into the action, we have

$$
\begin{aligned}
\pi\delta\Omega_2 =& d\left(\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 X^\mu - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2 X^\mu\right) && (39)\\
&+\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 + \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 && (40)\\
&-\bar{\varepsilon}^1\Gamma^\mu d\Theta^1\bar{\Theta}^2\Gamma_\mu d\Theta^2 - \bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 && (41)\\
=& d\left(\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 X^\mu - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2 X^\mu\right) && (42)\\
&+\bar{\Theta}^1\Gamma_\mu d\Theta^1\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^1\Gamma^\mu d\Theta^1 - \bar{\Theta}^2\Gamma_\mu d\Theta^2\bar{\varepsilon}^2\Gamma^\mu d\Theta^2 - \bar{\varepsilon}^1\Gamma^\mu d\Theta^1\bar{\Theta}^2\Gamma_\mu d\Theta^2 && (43)\\
=& d\left(\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 X^\mu - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2 X^\mu\right) && (44)\\
&-(\bar{\varepsilon}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1)d\sigma^\alpha d\sigma^\beta + (\bar{\varepsilon}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2)d\sigma^\alpha d\sigma^\beta && (45)\\
&+(\bar{\varepsilon}^2\Gamma^\mu\partial_\alpha\Theta^2\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2)d\sigma^\alpha d\sigma^\beta - (\bar{\varepsilon}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2)d\sigma^\alpha d\sigma^\beta && (46)\\
=& d\left(\bar{\varepsilon}^1\Gamma_\mu d\Theta^1 X^\mu - \bar{\varepsilon}^2\Gamma_\mu d\Theta^2 X^\mu\right) && (47)\\
&-(\bar{\varepsilon}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1)d\sigma^\alpha d\sigma^\beta + (\bar{\varepsilon}^2\Gamma^\mu\partial_\alpha\Theta^2\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2)d\sigma^\alpha d\sigma^\beta && (48)
\end{aligned}
$$

From the expression, we see that, we only need to consider the term like

$$\Xi = \bar{\varepsilon}\Gamma^\mu d\Theta\bar{\Theta}\Gamma_\mu d\Theta \tag{49}$$

This may be rewritten as

$$\Xi = (\Xi_1 + \Xi_2)\, d^2\sigma \tag{50}$$

where

$$
\begin{aligned}
\Xi_1 =& \frac{2}{3}\left(\bar{\varepsilon}\Gamma^\mu\dot{\Theta}\bar{\Theta}\Gamma_\mu\Theta' + \bar{\varepsilon}\Gamma^\mu\Theta'\dot{\bar{\Theta}}\Gamma_\mu\Theta + \bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}'\Gamma_\mu\dot{\Theta}\right)\\
\Xi_2 =& \frac{1}{3}\left(\bar{\varepsilon}\Gamma^\mu\dot{\Theta}\bar{\Theta}\Gamma_\mu\Theta' + \bar{\varepsilon}\Gamma^\mu\Theta'\dot{\bar{\Theta}}\Gamma_\mu\Theta - 2\bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}'\Gamma_\mu\dot{\Theta}\right)\\
=& \frac{1}{3}\frac{\partial}{\partial\tau}\left(\bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}\Gamma_\mu\Theta'\right) - \frac{1}{3}\frac{\partial}{\partial\sigma}\left(\bar{\varepsilon}\Gamma^\mu\Theta\bar{\Theta}\Gamma_\mu\dot{\Theta}\right)
\end{aligned}
$$

Notice that Ξ_2 is a total derivative, and Ξ_1 vanishes because it's of the form

$$\bar{\varepsilon}\Gamma_\mu\psi_{[1}\bar{\psi}_2\Gamma^\mu\psi_3$$

from the proof of Problem 2, we see it equals to zero. Thus the variation is a total derivative, thus the action is supersymmetric. □

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