Problem 1 Construct the conserved supersymmetry charges for open strings in the light-cone gauge formalism of Section 5.3 and verify that they satisfy the supersymmetry algebra.

Hint: the 16 supercharges are given by two eightcomponent spinors, Q^+ and Q^-. The Q^+ s anticommute to P^+, the Q^- s anticommute to P^-, and the anticommutator of Q^+ and Q^- gives the transverse momenta.

Solution.

Recall that the GS superstring action in light cone gauge has the following form,

$$S = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X_i \partial^\alpha X^i + \bar{S}^a \rho^\alpha \partial_\alpha S^a\right) \tag{1}$$

where $S^a = \left(S_1^a, S_2^a\right)$ is 16-component space-time spinor, and because it's type-I superstring, we have $S_1 = S_2$

There are two kinds of SUSYs, the first one preserve the light-gauge of S^a and the second one does not preserve the light-cone gauge of S^a (this corresponds to kappa symmetry).

Let's first consider the SUSY that preserve the light-cone gauge of S^a, i.e. such $\delta S^a = \varepsilon^a$ that $\Gamma^+\varepsilon = 0$. From the constraint, we see that corresponding SUSY transformations are

$$\delta S^a = \varepsilon^a, \quad \delta X^i = \bar{\varepsilon}\Gamma^i S = 0. \tag{2}$$

Using the familiar Noether theorem, we have

$$\delta S = -\frac{1}{2\pi}\int d^2\sigma \bar{\varepsilon}^a \rho^\alpha \partial_\alpha S^a = \int d^2\sigma \bar{\varepsilon}^a \partial_\alpha J^{a\alpha} \tag{3}$$

from which we obtain

$$\tilde{Q}^a = -2\pi \int_0^\pi d\sigma J_a^0 = \int_0^\pi d\sigma \rho^0 S_a, \tag{4}$$

where the zero modes of Noether currents are symmetry algebra generators,

$$Q^a = \sqrt{2p^+} S_0^a \tag{5}$$

It's easily checked that they indeed generate appropriate transformations: $\delta S^a = \sqrt{2p^+}\varepsilon^a, \quad \delta X^i = 0.$

Now let's consider then SUSY transformations that violate light-cone gauge of S^a. Such SUSY transformations should be accompanied with local κ -transformation $\delta S = \varepsilon + 2\Gamma \cdot \Pi_\alpha \kappa^\alpha$ in a way to make total transformation preserving light-cone gauge (preserving the action). The corresponding SUSY transformation can be constructed as

$$\delta S^a = \rho^\alpha \partial_\alpha X^i \Gamma^i_{ab} \bar{\varepsilon}^{\dot{b}} \sqrt{p^+}, \quad \delta X^i = \Gamma^i_{ab} \bar{\varepsilon}^{\dot{b}} S^a / \sqrt{p^+} \tag{6}$$

Here Γ -matrices play the role of Clebsh-Gordon coefficients, connecting three representations of Spin(8) for the aim described above:

$$|\dot{a}\rangle = \Gamma^i_{\dot{a}b} S_0^b |i\rangle, \quad |i\rangle = \Gamma^i_{\dot{a}b} S_0^b |\dot{a}\rangle. \tag{7}$$

Using the noether theorem and consider the zero mode part, we have

$$Q^{\dot{a}} = \frac{1}{\sqrt{p^+}} \Gamma^i_{\dot{a}b} \sum_n S^b_{-n} \alpha^i_n. \tag{8}$$

Supersymmetry generators satisfy anticommutation relations:

$$\left\{Q^a, Q^b\right\} = 2p^+ \delta^{ab}, \tag{9}$$

$$\left\{Q^a, Q^{\dot{a}}\right\} = \sqrt{2}\Gamma^i_{a\dot{a}} p^i, \tag{10}$$

$$\left\{Q^{\dot{a}}, Q^{\dot{b}}\right\} = 2H\delta^{\dot{a}\dot{b}} \tag{11}$$

where

$$H = \frac{1}{2p^+}\left(\left(p^i\right)^2 + 2N\right)$$

excitation number operator is

$$N = \sum_{m=1}^{\infty}\left(\alpha^i_{-m}\alpha^i_m + mS^a_{-m}S^a_m\right)$$

The first one (9) is obvious from the anti-commutator of $\left\{S^a_m, S^b_n\right\} = \delta_{m+n,0}\delta^{ab}$. For the second one (9)

$$\left\{Q^a, Q^{\dot{a}}\right\} = \sqrt{2}\Gamma^i_{\dot{a}b}\sum_n \{S^a_0, S^b_{-n}\}\alpha^i_n \tag{12}$$

$$= \sqrt{2}\Gamma^i_{\dot{a}b}\sum_n \delta_{-n,0}\delta^{ab}\alpha^i_n \tag{13}$$

$$= \sqrt{2}\Gamma^i_{a\dot{a}}p^i. \tag{14}$$

For the last one (11)

$$\left\{Q^{\dot{a}}, Q^{\dot{b}}\right\} = \frac{1}{p^+}\Gamma^i_{\dot{a}c}\Gamma^j_{\dot{b}d}\sum_{n,m}\left(S^c_{-n}S^d_{-m}\alpha^i_n\alpha^j_m + S^d_{-m}S^c_{-n}\alpha^j_m\alpha^i_n\right) \tag{15}$$

$$= \frac{1}{p^+}\Gamma^i_{\dot{a}c}\Gamma^j_{\dot{b}d}\sum_{n,m}\left(S^c_{-n}S^d_{-m}\alpha^i_n\alpha^j_m + S^d_{-m}S^c_{-n}\alpha^j_m\alpha^i_n + S^d_{-m}S^c_{-n}\alpha^i_n\alpha^j_m - S^d_{-m}S^c_{-n}\alpha^i_n\alpha^j_m\right) \tag{16}$$

$$= \frac{1}{p^+}\Gamma^i_{\dot{a}c}\Gamma^j_{\dot{b}d}\sum_{n,m}\left(\{S^c_{-n}, S^d_{-m}\}\alpha^i_n\alpha^j_m + S^d_{-m}S^c_{-n}[\alpha^j_m, \alpha^i_n]\right) \tag{17}$$

$$= 2H\delta^{\dot{a}\dot{b}} \tag{18}$$

This complete the proof. □

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