Weak Hopf symmetries behind $2d$ topological phases

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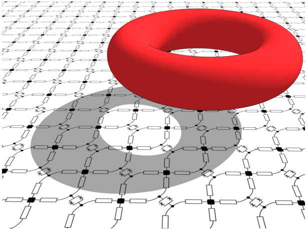
Overview

Topologically ordered phases

Anyon model: physical perspective

Anyon model:

- Topological charges;
- Fusion/splitting rule;
- Mutual statistics;
- Topological spin.



ENTANGLEMENT PATTERN!

Physics of topologically ordered phase (TOP):

- Topological order (compared with local order), phase transition beyond the Ginzburg–Landau spontaneous symmetry breaking (SSB) theory.
- Topologically protected ground state degeneracy;
- Topological entanglement entropy;
- Fractional charges/statistics;
- Boundary physics (edge states) and boundary-bulk duality;
- No TOP in $1d$! Rich in $2d$; higher dimensional case (not fully understood!).

Symmetry enriched topological (SET) phase and symmetry protected topological (SPT) phase.

	No TO	TO
No Sym	Trivial	TO
Sym	SPT	SET

SET are richer and need a more complicated mathematical characterization

- SPT: Haldane chain, topological insulator.
- SET: FQHE.
- Exist even in $1d$.
- Lattice models are hard to construct.

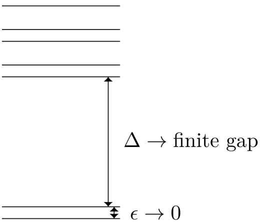
The mathematical theory of anyons (2d)

- Topological order (TO) is characterized by a unitary modular tensor category (UMTC).
- SPT and SET are characterized by (i) modular extension of UMTC or (ii) G-crossed modular tensor category.
- Anyon condensation theory.
- Boundary-bulk duality.
- Some notions: gapped/gapless, chiral/non-chiral, anomalous/anomaly-free.

Our main focus of this talk is the gapped anomaly-free non-chiral topological order (without symmetry)!

A rigorous definition of topological order at the Hamiltonian level?

- Gapless case: far from reaching!
- Gapped case: adiabatic path $H(\lambda)$ without closing the energy gap, $H(0) \simeq H(1)$.



Definition of gap.

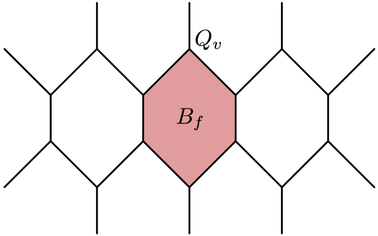
Hamiltonian theory of topological phase

- Topologically ordered state: Long-range entangled states.
- Topological order: a class of (local) gapped Hamiltonians $(H, \mathcal{H})$ which realize a TQFT.

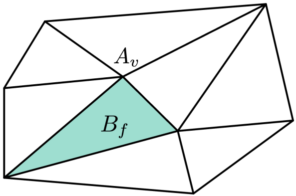
Lattice model of 2d topological order

Big picture

Relation between Kitaev quantum double model and Levin-Wen string-net model :



String-net (SN) model

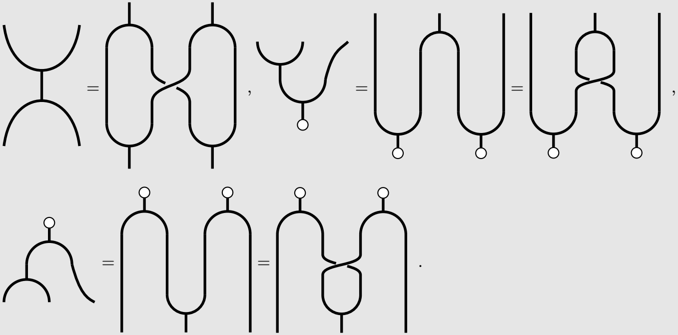


Quantum double (QD) model

- SN with input UFC ⇔ QD with input a connected WHA.
- Multifusion SN ⇔ QD with input a general WHA.
- Morita equivalence.
- Duality between different models. (LU equivalence!)
- EM duality.

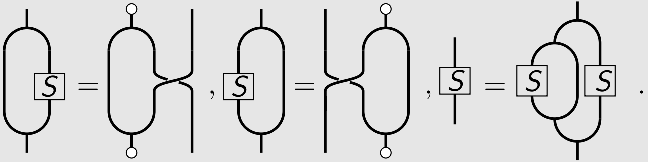
Weak bialgebra (WBA)

In a braided fusion category $\mathcal{C}$, a WBA is an object $W \in \mathcal{C}$ that equipped with the following structure (i) Algebra (W, μ, η) (ii) Coalgebra (W, μ, ε), such that



Weak Hopf algebra (WHA)

A WHA in $\mathcal{C}$ is a WBA equipped with an antipode $S : W \to W$ such that



Take $\mathcal{C} = \mathrm{Vect}_{\mathbb{C}}$, we obtain the complex WHA.

- $\varepsilon_L(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1_W)(h \otimes 1_W)) = \sum_{(1_W)} \varepsilon(1_W^{(1)} h) 1_W^{(2)}$ and we denote $W_L = \varepsilon_L(W)$;
- $\varepsilon_R(h) = (\mathrm{id} \otimes \varepsilon)((1_W \otimes h)\Delta(1_W)) = \sum_{(1_W)} 1_W^{(1)} \varepsilon(h 1_W^{(2)})$ and we denote $W_R = \varepsilon_R(W)$.

- The linear span J of the elements

$$\varphi \otimes xh - \varphi(x \rightharpoonup \varepsilon) \otimes h, \quad x \in W_L,$$
$$\varphi \otimes yh - \varphi(\varepsilon \leftharpoonup y) \otimes h, \quad y \in W_R,$$

is a two-sided ideal of $\hat{W}^{\mathrm{cop}} \otimes W$. We denote the quotient algebra $(\hat{W}^{\mathrm{cop}} \otimes W)/J$ as $D(W)$ and equivalent classes in $D(W)$ as $[\varphi \otimes h]$ for $\varphi \otimes h \in \hat{W}^{\mathrm{cop}} \otimes W$.

- J is used to ensure the operations below are well-defined and satisfy the axioms of WHA.

Quantum double of WHA

For a WHA $W \in \mathrm{Vect}_{\mathbb{C}}$, its quantum double is defined as the space $D(W) = (\hat{W} \otimes W)/J$ equipped with the following weak Hopf algebra structure:

(1) The multiplication $[\varphi \otimes h][\psi \otimes g] = \sum_{(\psi),(h)} [\varphi\psi^{(2)} \otimes h^{(2)} g] \langle \psi^{(1)}, S^{-1}(h^{(3)}) \rangle \langle \psi^{(3)}, h^{(1)} \rangle$.

(2) The unit $[\varepsilon \otimes 1_W]$.

(3) The comultiplication $\Delta([\varphi \otimes h]) = \sum_{(\varphi),(h)} [\varphi^{(2)} \otimes h^{(1)}] \otimes [\varphi^{(1)} \otimes h^{(2)}]$.

(4) The counit $\varepsilon([\varphi \otimes h]) = \langle \varphi, \varepsilon_R(S^{-1}(h)) \rangle$.

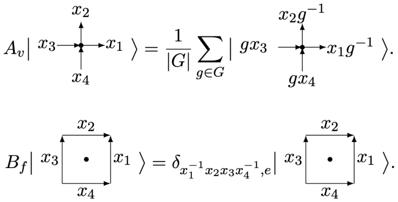
(5) The antipode $S([\varphi \otimes h]) = \sum_{(\varphi),(h)} [\hat{S}^{-1}(\varphi^{(2)}) \otimes S(h^{(2)})] \langle \varphi^{(1)}, h^{(3)} \rangle \langle \varphi^{(3)}, S^{-1}(h^{(1)}) \rangle$.

The quantum double has a canonical quasitriangular structure, ensuring that the representation category of $D(W)$ is braided.

Lattice model of 2d topological order

Weak Hopf quantum double model

Quantum double model is defined for finite group G:



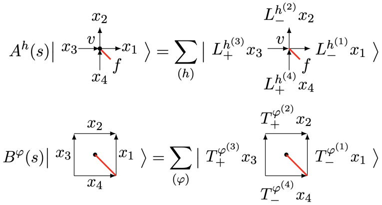
Haar integral

A left (resp. right) integral of W is an element l (resp. r) satisfying $xl = \varepsilon_L(x)l$ (resp. $rx = r\varepsilon_R(x)$). A left (resp. right) integral l (resp. r) is called left (resp. right) normalized if $\varepsilon_L(l) = 1_W$ (resp. $\varepsilon_R(r) = 1_W$). If h is both a left and right integral, it is called a two-side integral. A Haar integral in W (or Haar measure on $\hat{W}$) is a two-side normalized two-side integral.

For weak Hopf quantum double model $D(W)$:

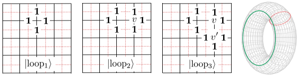
- W-action: $L_+^g|z\rangle = |gz\rangle$, $L_-^g|z\rangle = |zS^{-1}(g)\rangle$.
- $\hat{w}$-action:
  $T_+^{\varphi}|x\rangle = |\varphi \rightharpoonup x\rangle = |\sum_{(x)}\langle\varphi, x^{(2)}\rangle x^{(1)}\rangle$,
  $T_-^{\varphi}|x\rangle = |x \leftharpoonup \hat{S}(\varphi)\rangle =$
  $|\sum_{(x)}\langle\hat{S}(\varphi), x^{(1)}\rangle x^{(2)}\rangle$.
- Vertex operator $A_v^h = L^{h^{(1)}} \otimes \cdots \otimes L^{h^{(n)}}$.
- Face operator $B_f^{\varphi} = T^{\varphi^{(1)}} \otimes \cdots \otimes T^{\varphi^{(n)}}$.

Haar integral $h \in W$ and $\varphi \in \hat{W}$.



Lattice model of 2d topological order

Weak Hopf quantum double model



- The bulk topological phase is characterized by the UMTC $\mathrm{Rep}(D(W))$.
- W is bulk gauge symmetry, $D(W)$ is bulk charge symmetry.
- Ribbon operators create topological excitations.
- Boundary gauge symmetry is W-comodule algebra, boundary charge symmetry is a WHA.
- The boundary phase is a UFC $\mathcal{B}$, the boundary-bulk duality is given by $\mathrm{Rep}(D(W)) \simeq \mathcal{Z}(\mathcal{B})$.
- Connection with string-net model?

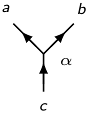
Lattice model of 2d topological order

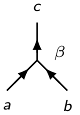
Multifusion string-net model

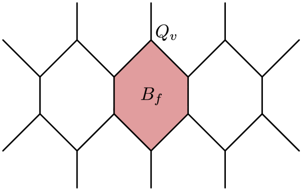
String-net

A connected oriented trivalent lattice Σ is called a string-net. For a string-net model with an input UMFC $\mathcal{D}$, its topological excitation is given by UMTC $\mathcal{Z}(\mathcal{D})$.

Vertex space $\mathcal{H}_v$: (gauge choice $Y_c^{ab} = (d_a d_b / d_c)^{1/2}$)

$(Y_c^{ab})^{-1/2}$  $= |c \to a, b; \alpha\rangle$,

$(Y_c^{ab})^{-1/2}$  $= \langle a, b \to c; \beta|$.

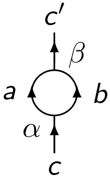


String-net (SN) model

Total space $\mathcal{H}_{\mathrm{tot}} = \otimes_v \mathcal{H}_v$.

Lattice model of 2d topological order

Multifusion string-net model: Topological local move

- loop move: 

$= \delta_{c,c'}\delta_{\alpha,\beta} Y_c^{ab}$ 

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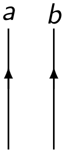
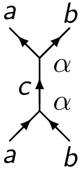
- parallel move: 

$= \sum_{c,\alpha} \frac{1}{Y_c^{ab}}$ 

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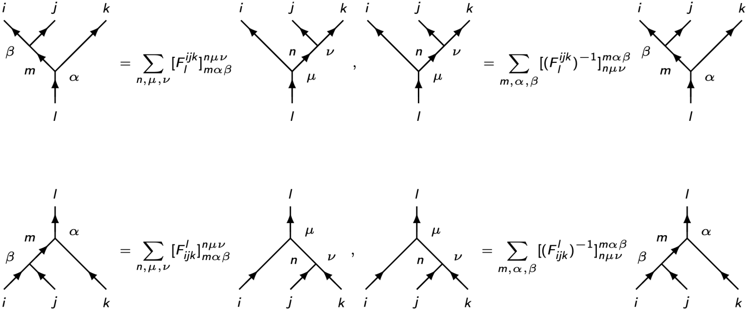
Lattice model of 2d topological order

Multifusion string-net model: Topological local move

- F-move



Topological local move

The loop move, parallel move, and F-move, collectively known as topological local moves, are equivalent to the Pachner moves.

Lattice model of 2d topological order

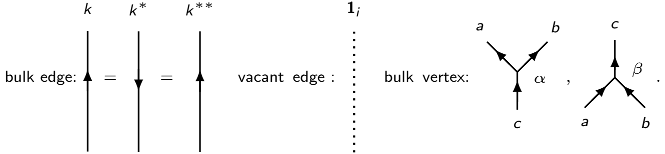
Multifusion string-net model

Input data of multifusion string-net

The input data for the generalized multifusion string-net are:

1. String type: $\mathrm{Irr}(\mathcal{D})$;
2. Fusion rule: $N_{a,b}^{c}$ (Recall that quantum dimensions d_a's are determined by the fusion rule);
3. Local normalization factor: Y_c^{ab}.
4. F-symbols: F_l^{ijk}, F_{ijk}^{l}.

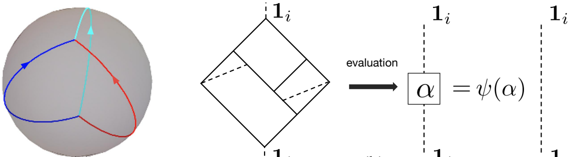
- A fully labeled string-net



Lattice model of 2d topological order

Multifusion string-net model

- The multifusion category $\mathcal{D} = \oplus_{i,j\in I}\mathcal{D}_{i,j}$, $\mathbf{1} = \oplus_{i\in I}\mathbf{1}_i$. $X_{i,j} \otimes Y_{j,k} \in \mathcal{D}_{i,k}$.
- For weak Hopf algebra W, its representation category $\mathrm{Rep}(W)$ is a multifusion category.
- String-net ground state



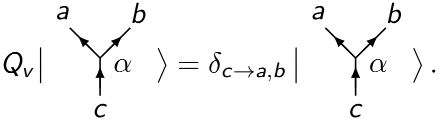
(a) (b)

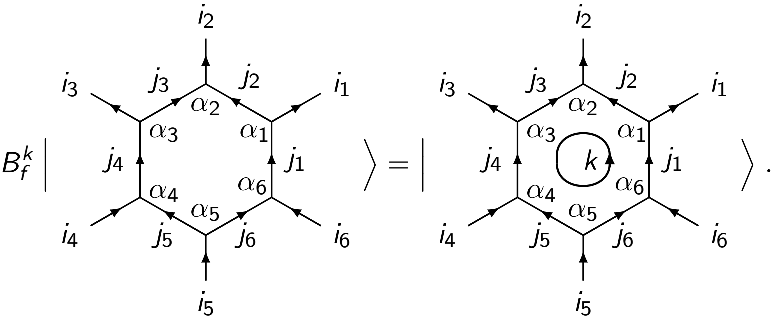
Ground state $|\psi\rangle = \sum_\alpha \psi(\alpha)|\alpha\rangle$. The coefficient is calculated by evaluation.

Lattice model of 2d topological order

Multifusion string-net model

String-net lattice model (vertex and face operators)





String-net lattice model

The local stabilizers Q_v and B_f functions are projectors and mutually commute. Consequently, the Hamiltonian of the generalized multifusion string-net ($J_v, J_f > 0$, $B_f = \sum_{k \in \mathrm{Irr}(\mathcal{D})} w_k B_f^k$, $w_k = Y_{\mathbf{1}}^{k^* k} / \sum_{l \in \mathrm{Irr}(\mathcal{D})} d_l^2$):

$$H = -J_v \sum_v Q_v - J_f \sum_f B_f$$

being a local commutative projector (LCP) Hamiltonian, exhibits a gap in the thermodynamic limit.

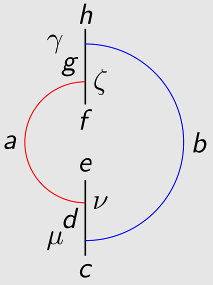
Multifusion string-net model

- For input UMFC $\mathcal{D}$, the topological excitation is given by UMTC $\mathcal{Z}(\mathcal{D})$ (Drinfeld center).
- The bulk gauge symmetry and charge symmetry are weak Hopf algebras.
- The boundary gauge symmetry is a W-comodule algebra, the boundary charge symmetry is a weak Hopf algbra.
- The domain wall gauge symmetry is a $W_1|W_2$-comodule algebra, the domain wall charge symmetry is a weak Hopf algebra.

DEFECTIVE LEVIN-WEN STRING-NET CAN BE REGARDED AS A MULTIFUSION STRING-NET!

Definition (Bulk tube algebra Tube($_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$))

The bulk tube algebra **Tube**($_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}$) is spanned by the following basis (up to planar isotopy):

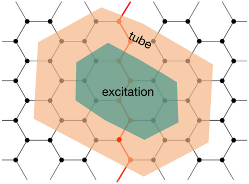


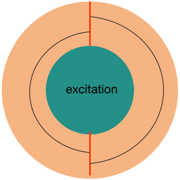
$$\left\{ \quad : \quad a, \cdots, h \in \mathrm{Irr}(\mathcal{D}), \mu, \nu, \gamma, \zeta \in \mathrm{Hom}_{\mathcal{D}} \right\}.$$

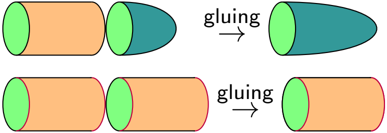
Note that the arrows have been omitted to avoid clutter in the equation; all edges are assumed to be directed upwards.

Weak Hopf tube algebra

Bulk tube algebra: Algebra structure

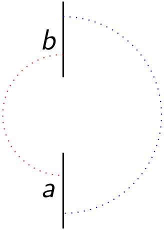






- The unit is given by

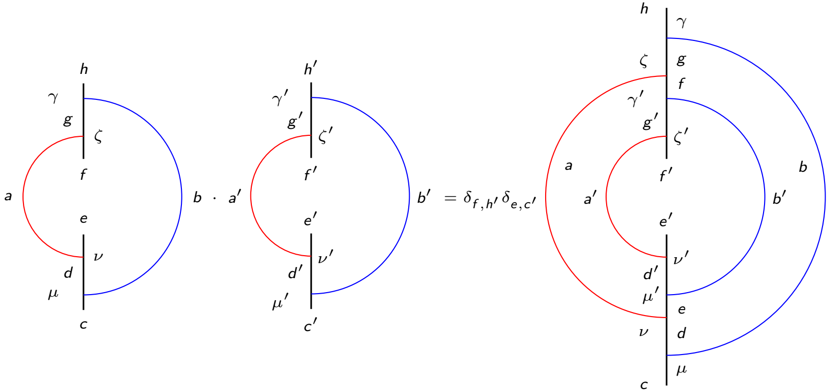
$$1 = \sum_{a,b}$$



Weak Hopf tube algebra

Bulk tube algebra: Algebra structure

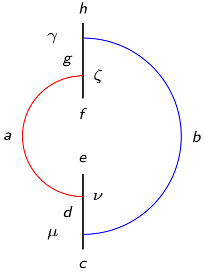
- The multiplication is of the form

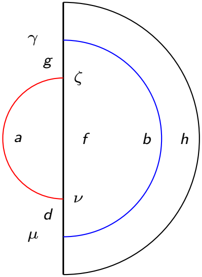


Weak Hopf tube algebra

Bulk tube algebra: Coalgebra structure

- The counit is of the form:

$\varepsilon \Bigg($ 

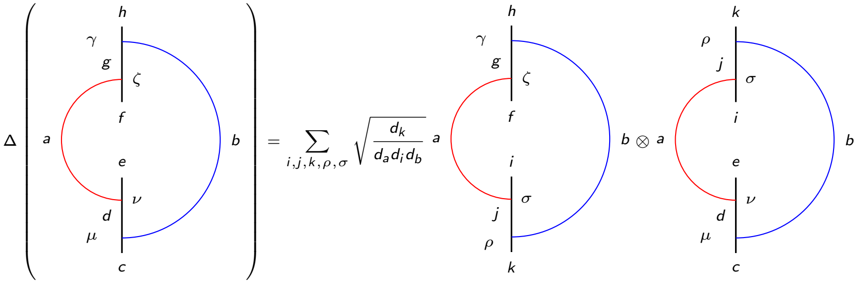
$\Bigg) = \frac{\delta_{e,f}\delta_{c,h}}{d_h}$ 

$= \delta_{e,f}\delta_{c,h}\delta_{d,g}\delta_{\nu,\zeta}\delta_{\mu,\gamma}\sqrt{\frac{d_a d_f d_b}{d_h}}$

Weak Hopf tube algebra

Bulk tube algebra: Coalgebra structure

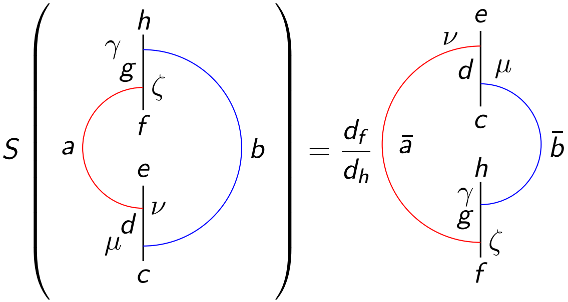
- The comultiplication is given by



Weak Hopf tube algebra

Bulk tube algebra: Antipode morphism

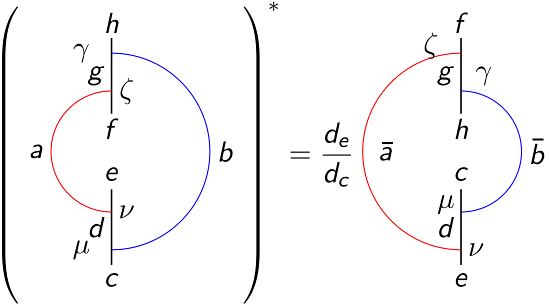
- Antipode operation S:



Weak Hopf tube algebra

Bulk tube algebra: C^* structure

- The $*$-operation is given by



Weak Hopf tube algebra

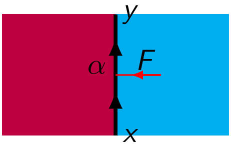
Bulk tube algebra: weak Hopf symmetry

Weak Hopf tube algebra

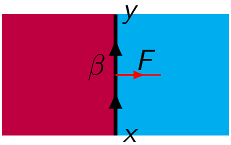
For any given UMFC $\mathcal{D}$, the tube algebra $\textbf{Tube}({}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}})$ is a C^* weak Hopf algebra.

For any $F \in \mathrm{Fun}_{\mathcal{D}|\mathcal{D}}(\mathcal{D}, \mathcal{D})$, we can construct a module V_F over the tube algebra.

$$V_F := \bigoplus_{x,y \in \mathrm{Irr}({}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}})} \mathrm{Hom}_{{}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}}(F(x), y),$$
$$V^F := \bigoplus_{x,y \in \mathrm{Irr}({}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}})} \mathrm{Hom}_{{}_{\mathcal{D}}\mathcal{D}_{\mathcal{D}}}(x, F(y)).$$



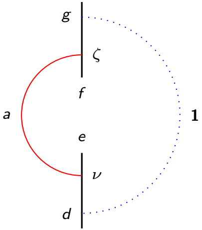
$\in V_F,$



$\in V^F,$

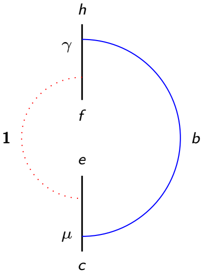
- Topological excitation are modules over tube algebra.
- Weak Hopf tube algebra is the charge symmetry for multifusion string-net.

$$\mathbf{L}({}_{\mathcal{D}}\mathcal{D}) = \mathrm{span}\left\{\right.$$



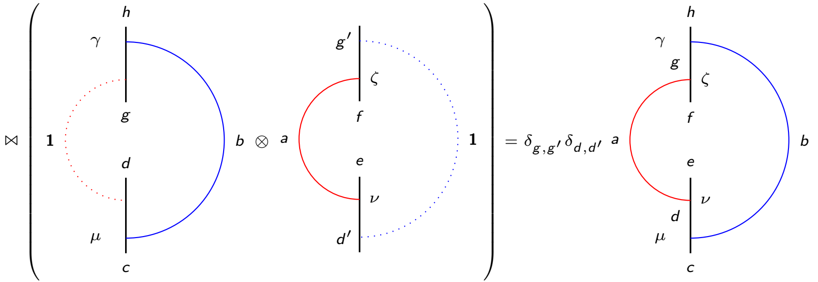
$$\left. : \quad a, \cdots, g \in \mathrm{Irr}(\mathcal{D}), \nu, \zeta \in \mathrm{Hom}_{\mathcal{D}} \right\}.$$

$$\mathbf{R}(\mathcal{D}_{\mathcal{D}}) = \mathrm{span}\left\{\right.$$



$$\left. : \quad b, \cdots, h \in \mathrm{Irr}(\mathcal{D}), \mu, \gamma, \in \mathrm{Hom}_{\mathcal{D}} \right\}.$$

Bulk tube algebra as crossed product of boundary tube algebra:



Boundary tube algebra

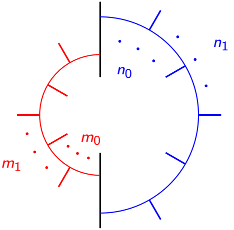
- The boundary tube algebra is a weak Hopf algebra.
- The boundary tube algebra can be regarded as the gauge symmetry of the bulk.

Weak Hopf tube algebra

Morita theory

General tube space $\mathbf{T}^{m_0,m_1;n_0,n_1}$ spanned by the tube string-net configurations :

$$\mathbf{T}^{m_0,m_1;n_0,n_1} = \mathrm{span}\left\{ \right.$$



$$\left. : \text{edge} \in \mathrm{Irr}, \text{vertex} \in \mathrm{Hom} \right\}$$

Morita equivalence

The tube space $\mathbf{T}^{m,m;n,n}$ are algebras for all $m, n \in \mathbb{N}$. The tube space $\mathbf{T}^{m,s;n,t}$ forms a right-$\mathbf{T}^{m,m;n,n}$ and left-$\mathbf{T}^{s,s;t,t}$ bimodule. These structures form a Morita context in the sense that

$$\mathbf{T}^{m,s;n,t} \otimes_{\mathbf{T}^{m,m;n,n}} \mathbf{T}^{s,m;t,n} \cong \mathbf{T}^{s,s;t,t}, \quad \mathbf{T}^{s,m;t,n} \otimes_{\mathbf{T}^{s,s;t,t}} \mathbf{T}^{m,s;n,t} \cong \mathbf{T}^{m,m;n,n}.$$

Thus $\mathbf{T}^{m,m;n,n}$'s are Morita equivalent for all $m, n \in \mathbb{N}$.

Weak Hopf symmetry behind $2d$ non-chiral topological order

- There are two type of weak Hopf symmetries for non-chiral topological phase: weak Hopf gauge symmetry and weak Hopf charge symmetry
- The weak Hopf is not unique, they are related by categorical Morita equivalence.
- For (weak Hopf) quantum double model, the weak Hopf gauge symmetry is the input weak Hopf algebra W, the weak Hopf charge symmetry is its quantum double $D(W)$.
- For (multifusion) string-net model, the weak Hopf gauge symmetry is given by boundary tube algebra, the weak Hopf charge symmetry is given by the bulk tube algebra.

Open problems

- Higher dimensional model and higher category structure.
- Entanglement property, entangle entropy is sensitive to defect and boundary.
- Weak Hopf quantum double $\Leftrightarrow$ extended string-net model.
- SET/SPT generalization of quantum double model and string-net model.
- Operator algebra perspective: stability, Haag duality, infinite-volume sector, etc.

THANK YOU FOR YOUR ATTENTIONS