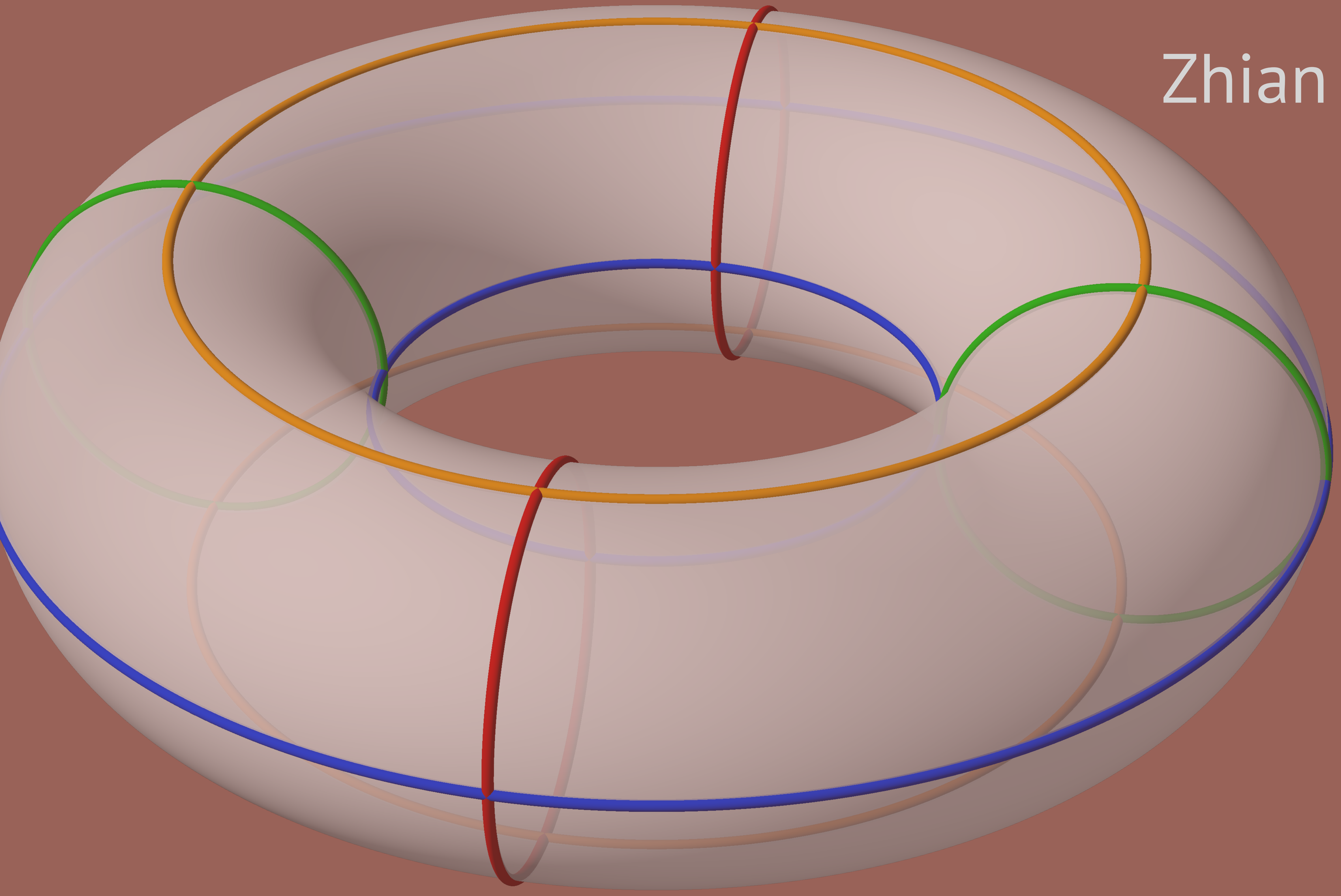
Quantum Information, Quantum Matter and Quantum Gravity

Gapped boundary theory of Hopf and weak Hopf quantum double model



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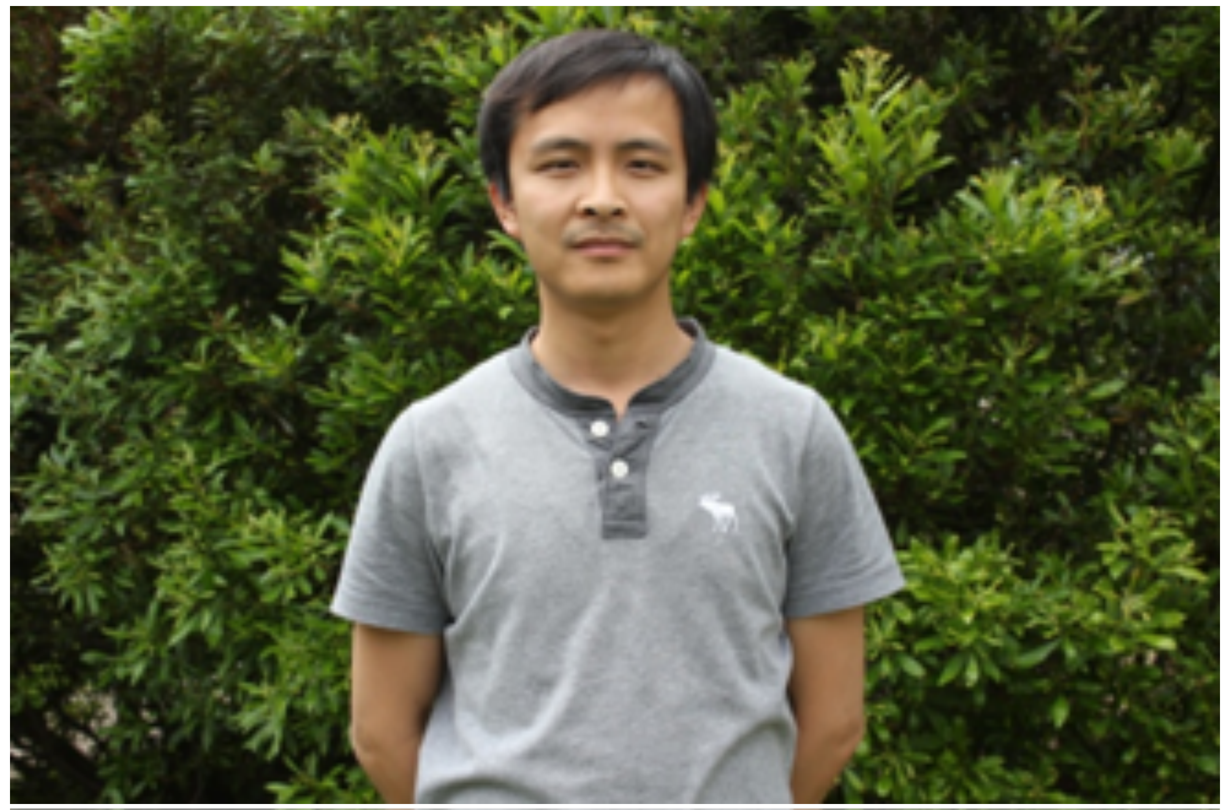

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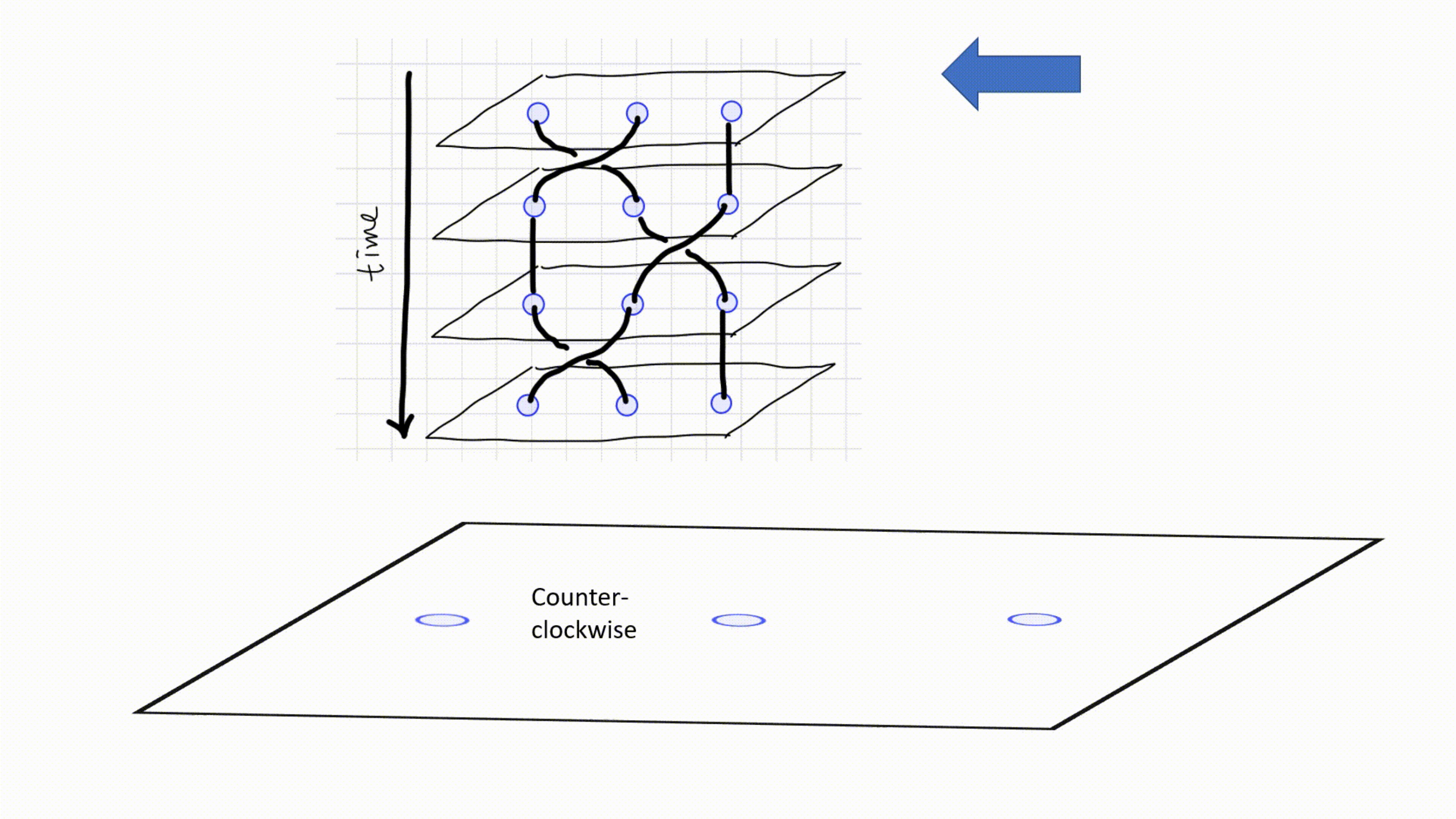


https://yk-liu.github.io/2019/Introduction-to-QC-and-TQC-Anyon-Model/

Electric-magnetic duality and Z2 symmetry enriched Abelian lattice gauge theory
Z Jia, D Kaszlikowski, S Tan,
arXiv preprint arXiv:2201.12361

Boundary and domain wall theories of 2d generalized quantum double model
Z Jia, D Kaszlikowski, S Tan
JHEP 2023 (7-160), 1-78; arXiv:2207.03970

On weak Hopf symmetry and weak Hopf quantum double model
Z Jia, S Tan, D Kaszlikowski, L Chang
Comm. Math. Phys. 402, 3045–3107; arXiv:2302.08131

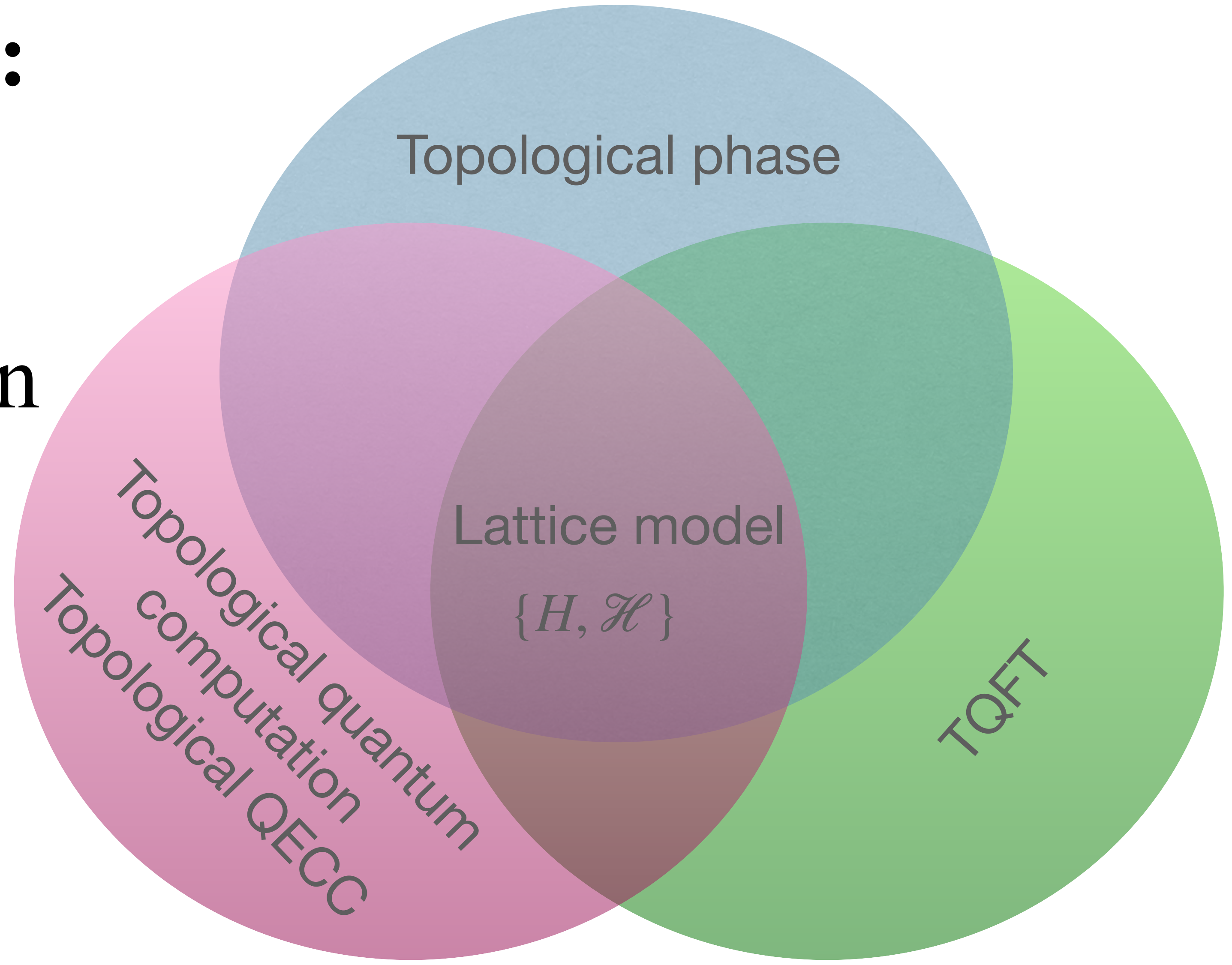
Quantum double model: from toric code to weak Hopf lattice gauge theory, Z Jia, S Tan , to be published

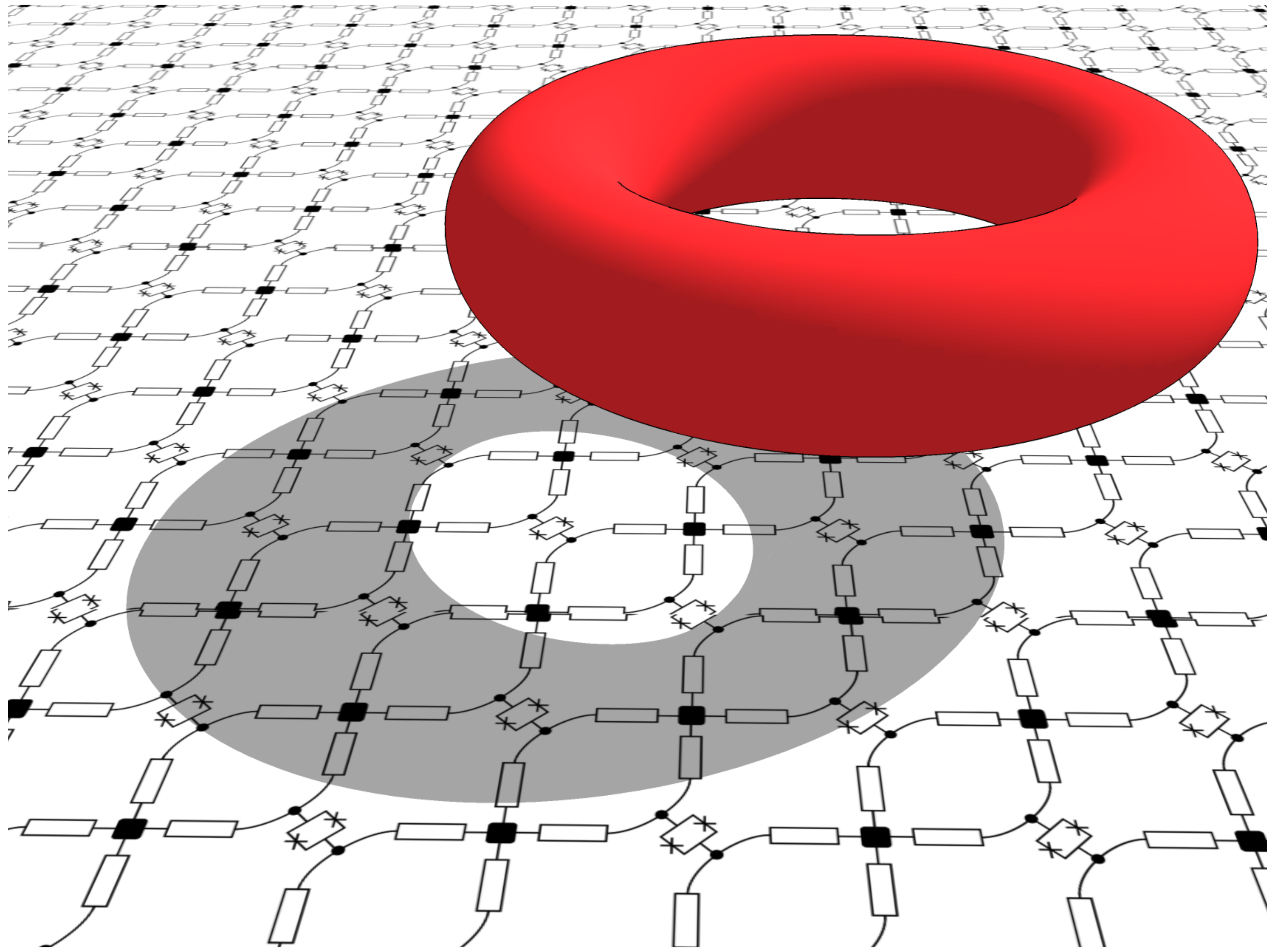
Topological order

Definition
Equivalence classes of gapped local Hamiltonian whose low-energy effective theories realize given TQFT

Physical properties:
- Topologically protected GSD
- Long-range entanglement
- Topological entanglement entropy
- Fractional statistics

(2+1)D anyon model:
- Topological charge
- Antiparticle
- Quantum dimension
- Fusion rule
- Braiding
- Topological spin
- S-matrix, T-matrix
- Verlinde formula





https://www.quantumtheory.nat.fau.eu/2017/04/21/superconducting-quantum-simulator-for-topological-order-and-the-toric-code/

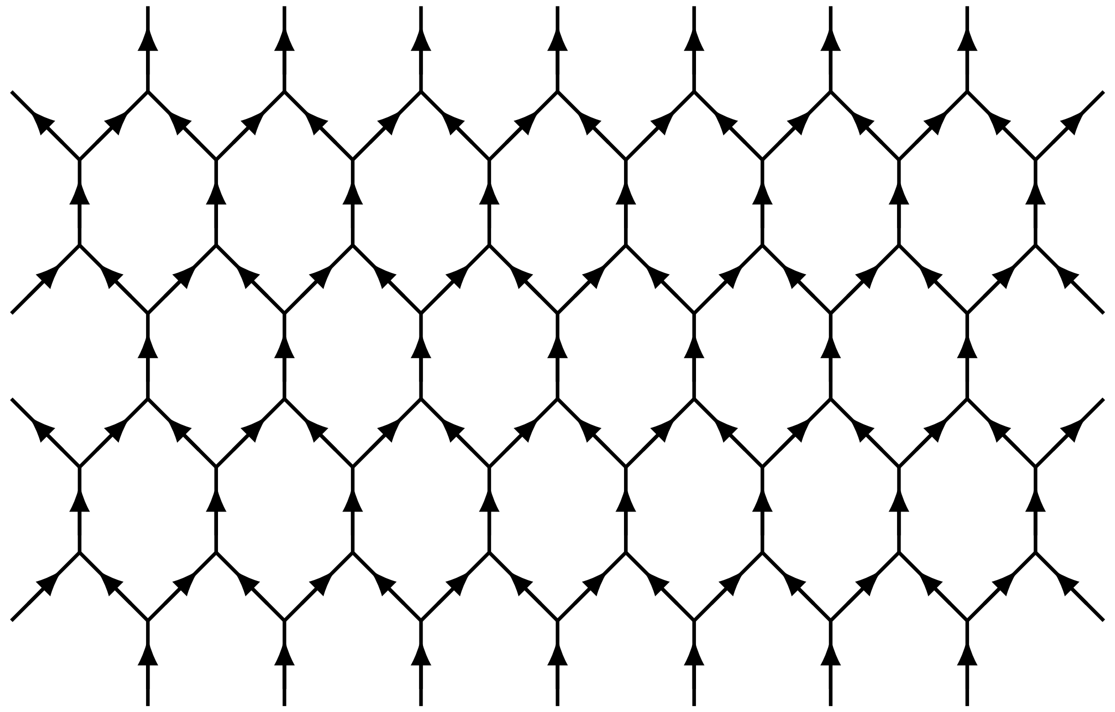
Topological order: lattice model

Definition
Equivalence classes of gapped local Hamiltonian whose low-energy effective theories realize given TQFT

The mathematical structure behind the topological order is **tensor category theory,** for the 2d case, the topological excitation is described by a unitary modular tensor category (UMTC)

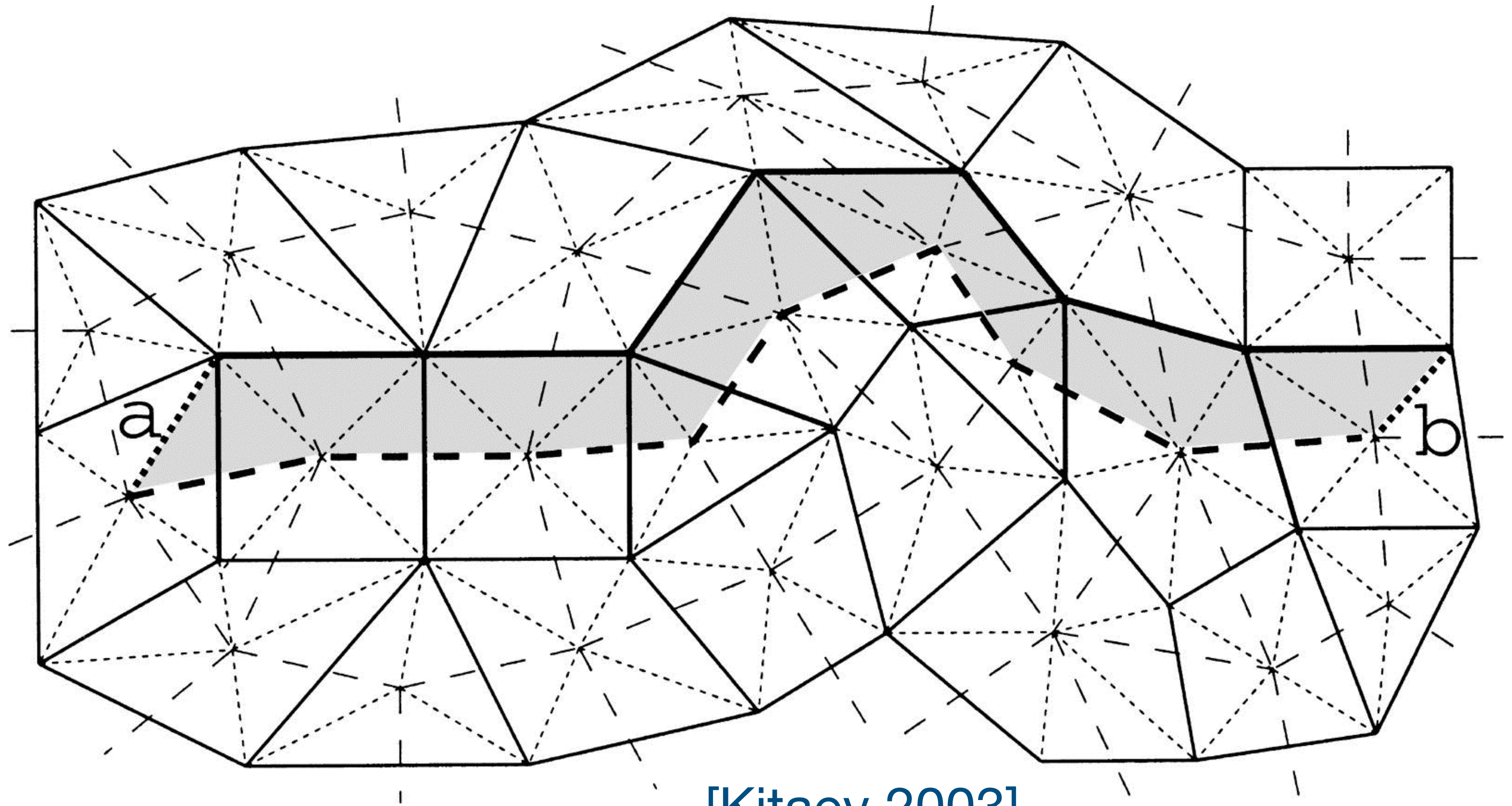
- anomaly-free
- non-chiral

Higher category for higher dimensional cases!



[Levin-Wen 2005]

Levin-Wen string-net model ↔ Turaev-Viro-Barrett-Westbury



[Kitaev 2003]

Kitaev quantum double model ↔ Dijkgraaf-Witten

Main topic of This talk

Hopf QD model: definition of Hopf algebra

A (complex) Hopf algebra is a complex vector space H equipped with several structure morphisms: multiplication $\mu : H \otimes H \to H$, unit $\eta : \mathbb{C} \to H$, comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to \mathbb{C}$ and antipode $S : H \to H$, for which some consistency conditions are satisfied:

1. (H, μ, η) is an algebra: $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$, and $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$.

2. (H, Δ, ε) is a coalgebra: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$.

3. $(H, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra: Δ and ε are algebra homomorphisms (equivalently μ and η are coalgebra homomorphisms).

4. The antipode S satisfies: $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta$.

Sweedler's Notation: $\Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)}; f \rightharpoonup h = \sum_{(h)} h^{(1)} f(h^{(2)})$

Hopf QD model: quantum double of Hopf algebra

The quantum double of H is the vector space $(H^\vee)^{\mathrm{cop}} \otimes H$ equipped with a Hopf algebra structure; we denote it as $D(H) = (H^\vee)^{\mathrm{cop}} \bowtie H$. The multiplication is given by

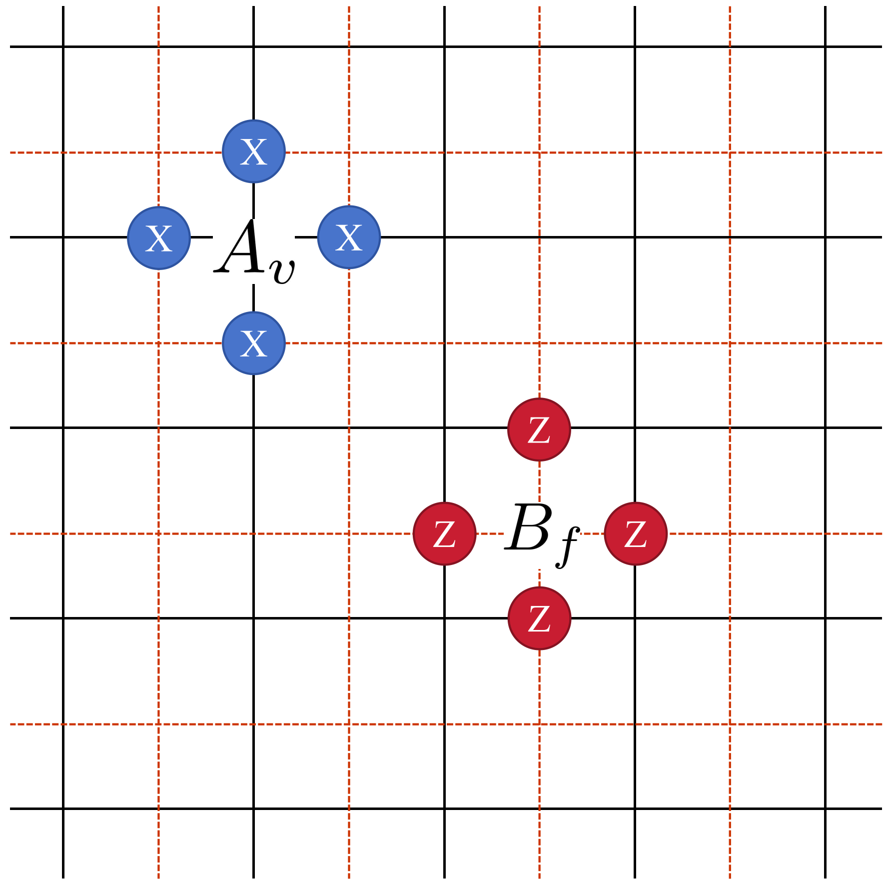
$$(\varphi \otimes x)(\psi \otimes y) := \sum_{(x)} \varphi\psi(S^{-1}(x^{(3)}) \bullet x^{(1)}) \otimes x^{(2)} y,$$

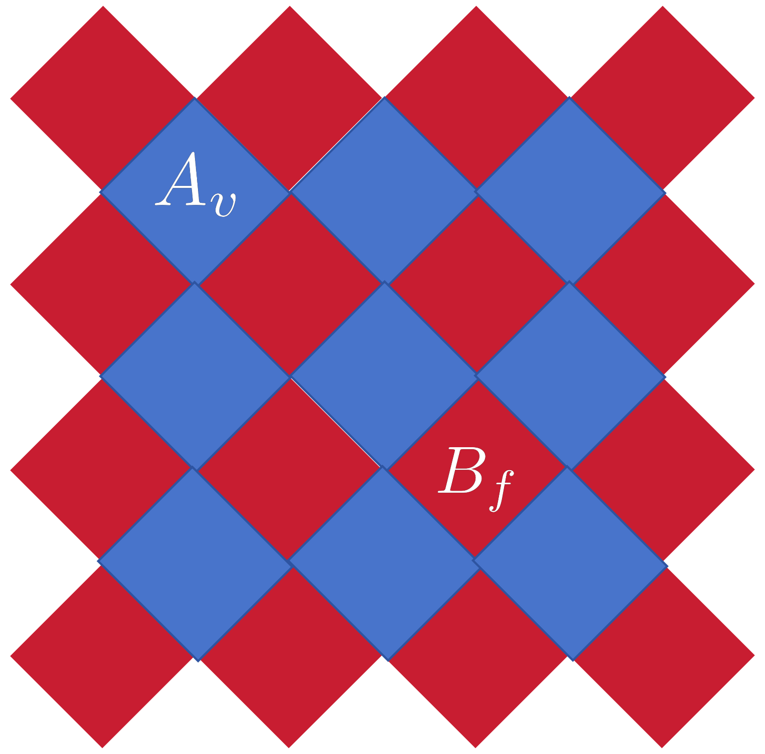
where "$\bullet$" denotes the argument of the function. The other data are given by

$$\begin{aligned}
1_{D(H)} &= \hat{1} \otimes 1,\\
\Delta_{D(H)}(\varphi \otimes x) &= \sum_{(\varphi)} \sum_{(x)} (\varphi^{(2)} \otimes x^{(1)}) \otimes (\varphi^{(1)} \otimes x^{(2)}),\\
\varepsilon_{D(H)}(\varphi \otimes x) &= \varepsilon(x)\varphi(1_H),\\
S_{D(H)}(\varphi \otimes x) &= \sum_{(\varphi)} \sum_{(x)} \langle \varphi^{(1)} \otimes \varphi^{(3)}, h^{(3)} \otimes S^{-1}(h^{(1)}) \rangle \hat{S}^{-1}(\varphi^{(2)}) \otimes S(h^{(2)}).
\end{aligned}$$

Hopf QD model: finite group algebra

Let G be a finite group, the linear span $\mathbb{C}[G]$ is a Hopf algebra
- Multiplication: $g \cdot h$
- Unit: 1_G
- Comultiplication: $\Delta(g) = g \otimes g, \forall g$
- Counit: $\varepsilon(g) = 1, \forall g$
- Antipode: $S(g) = g^{-1}, \forall g$

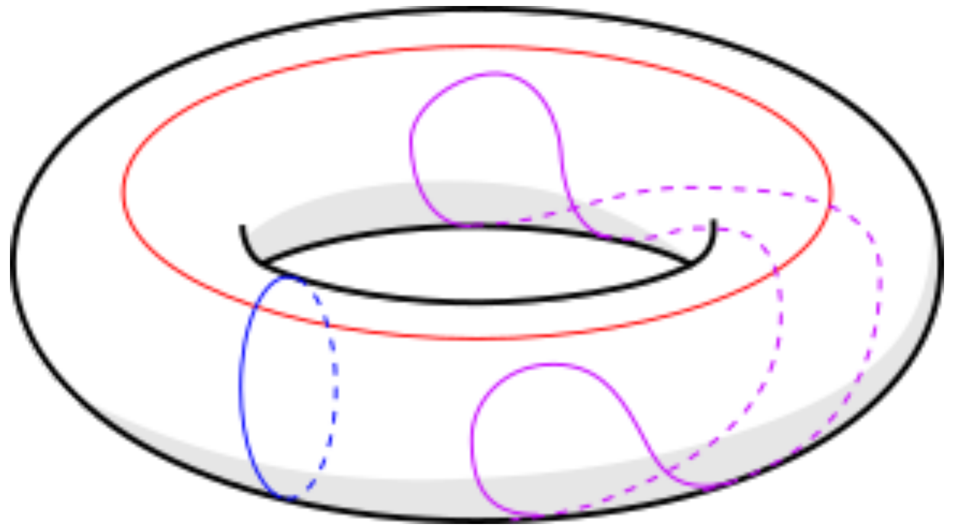


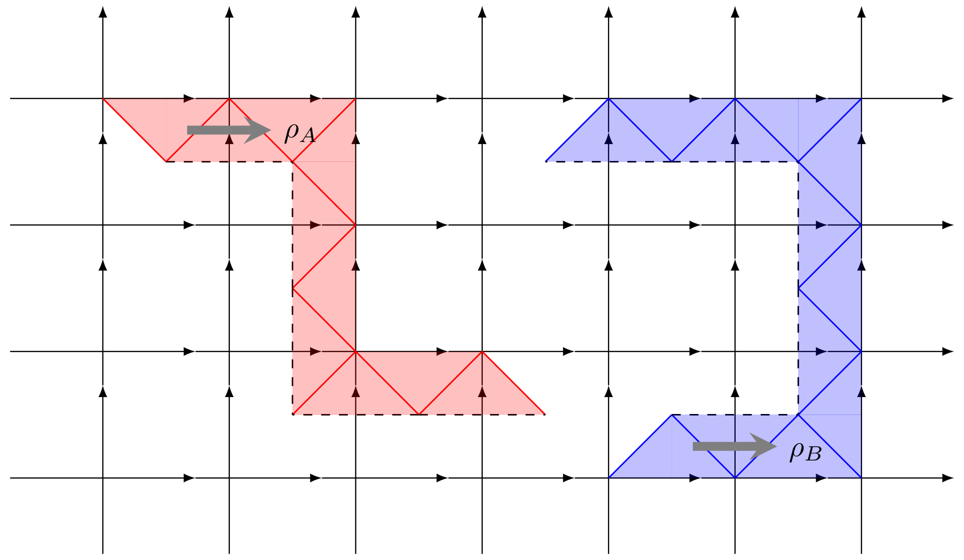


Kitaev quantum double model

$$A_v \left| \begin{array}{ccc} & x_2 & \\ x_3 & \cdot & x_1 \\ & x_4 & \end{array} \right\rangle = \frac{1}{|G|} \sum_{g \in G} \left| \begin{array}{ccc} & x_2 g^{-1} & \\ g x_3 & \cdot & x_1 g^{-1} \\ & g x_4 & \end{array} \right\rangle.$$

$$B_f \left| \begin{array}{ccc} & x_2 & \\ x_3 & \bullet & x_1 \\ & x_4 & \end{array} \right\rangle = \delta_{x_1^{-1} x_2 x_3 x_4^{-1}, e} \left| \begin{array}{ccc} & x_2 & \\ x_3 & \bullet & x_1 \\ & x_4 & \end{array} \right\rangle.$$





Many applications
- Topological QC
- Topological quantum memory
- Topological order
- Lattice gauge theory

Hopf QD model: general C* Hopf algebra

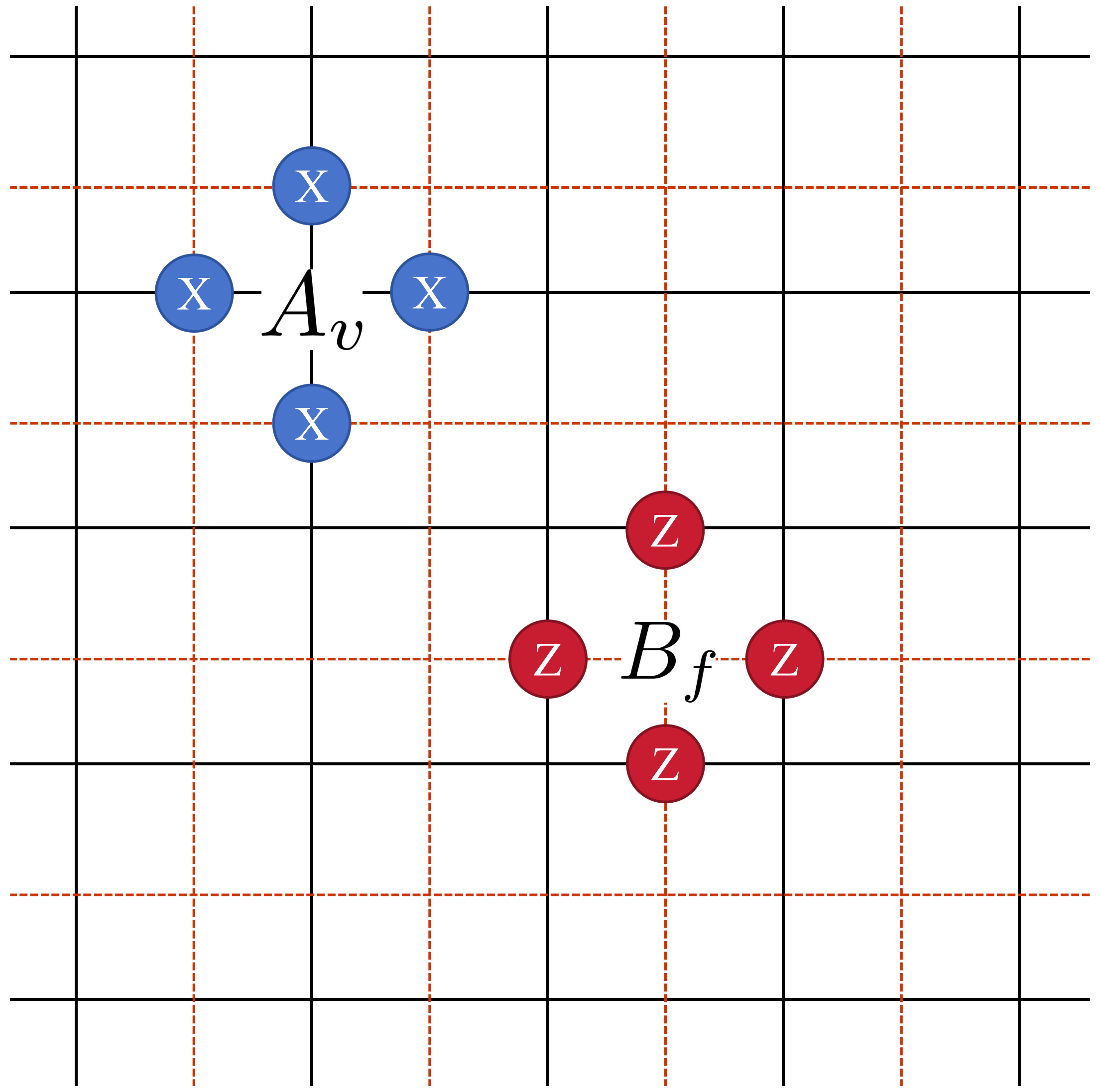
- Edge space: Hopf algebra H

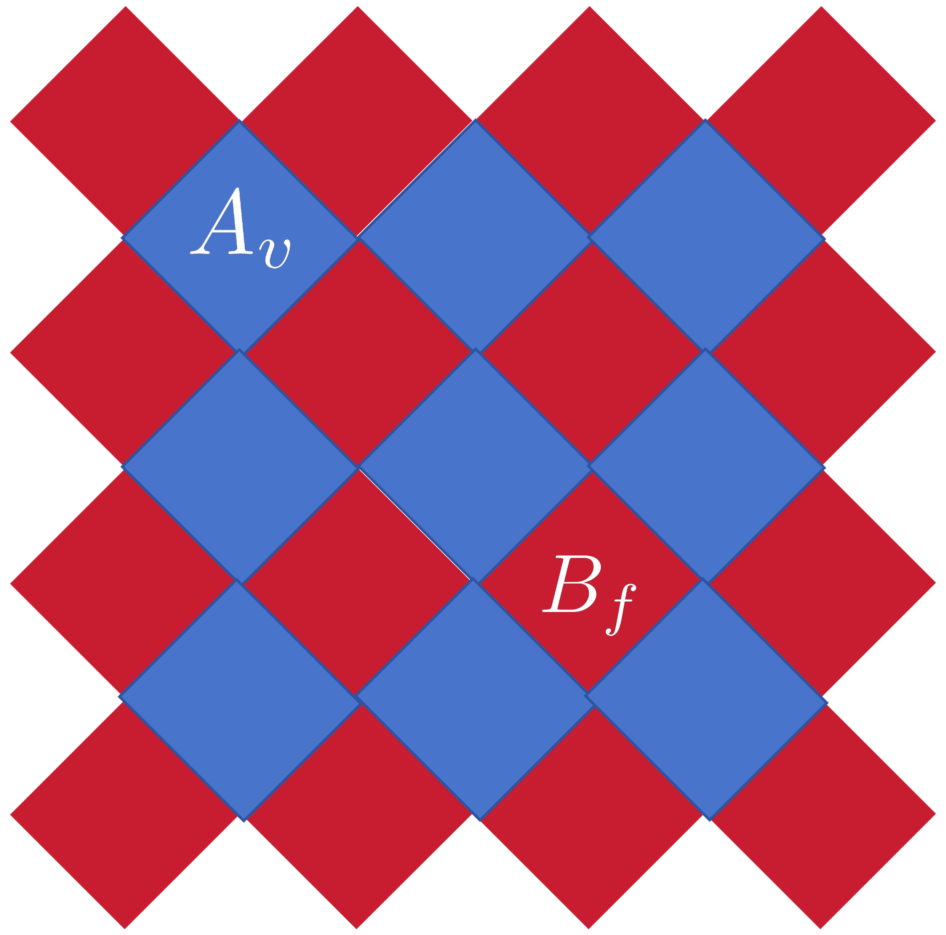
- Vertex: H as a left H-module:

$$L_+^h|x\rangle = |h \triangleright x\rangle = |hx\rangle,$$
$$L_-^h|x\rangle = |x \triangleleft S(h)\rangle = |xS(h)\rangle,$$

- Face: H as a left $\hat{H}$-module

$$T_+^\varphi|x\rangle = |\varphi \rightharpoonup x\rangle = |\sum_{(x)}\langle\varphi, x^{(2)}\rangle x^{(1)}\rangle,$$
$$T_-^\varphi|x\rangle = |x \leftharpoonup \hat{S}(\varphi)\rangle = |\sum_{(x)}\langle\hat{S}(\varphi), x^{(1)}\rangle x^{(2)}\rangle = |\sum_{(x)}\langle\varphi, S(x^{(1)})\rangle x^{(2)}\rangle$$





To construct the face and vertex operators, we need a new concept: **Haar integral of Hopf algebra**

Hopf QD model: general C* Hopf algebra

Haar integral

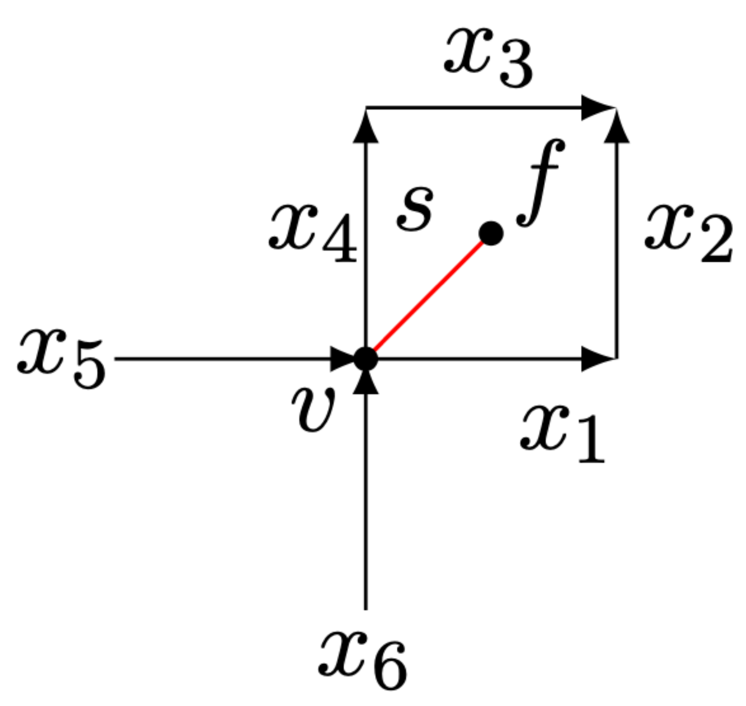
Let H be a Hopf algebra. A left (resp. right) integral of H is an element $\ell \in H$ (resp. $r \in H$) such that $x\ell = \varepsilon(x)\ell$ (resp. $rx = r\varepsilon(x)$) for all $x \in H$. They are normalized if $\varepsilon(\ell) = \varepsilon(r) = 1$. A two-sided integral is an element that is simultaneously a left and a right integral. A Haar integral is a normalized two-sided integral.

The Haar integral for the group algebra $\mathbb{C}[G]$ is $h_{\mathbb{C}[G]} = \frac{1}{|G|}\sum_{g\in G} g$, and the Haar measure is δ_{1_G}, where for any $g \in G$ the function $\delta_g : \mathbb{C}[G] \to \mathbb{C}$ is given by $\delta_g(h) = \delta_{g,h} = 1$ if $g = h$ and 0 otherwise. Hence the inner product on $\mathbb{C}[G]$ induced by δ_{1_G} is $\langle g, h\rangle = \delta_{g,h}$, for $\forall\, g, h \in G$.

Every C* Hopf algebra has a unique Haar integral !

Hopf QD model: general C* Hopf algebra

Local stabilizers



$$A^h(s) = \sum_{(h)} L_-^{h^{(1)}}(j_4) \otimes L_+^{h^{(2)}}(j_5) \otimes L_+^{h^{(3)}}(j_6) \otimes L_-^{h^{(4)}}(j_1),$$

$$B^\varphi(s) = \sum_{(\varphi)} T_-^{\varphi^{(1)}}(j_1) \otimes T_-^{\varphi^{(2)}}(j_2) \otimes T_+^{\varphi^{(3)}}(j_3) \otimes T_+^{\varphi^{(4)}}(j_4),$$

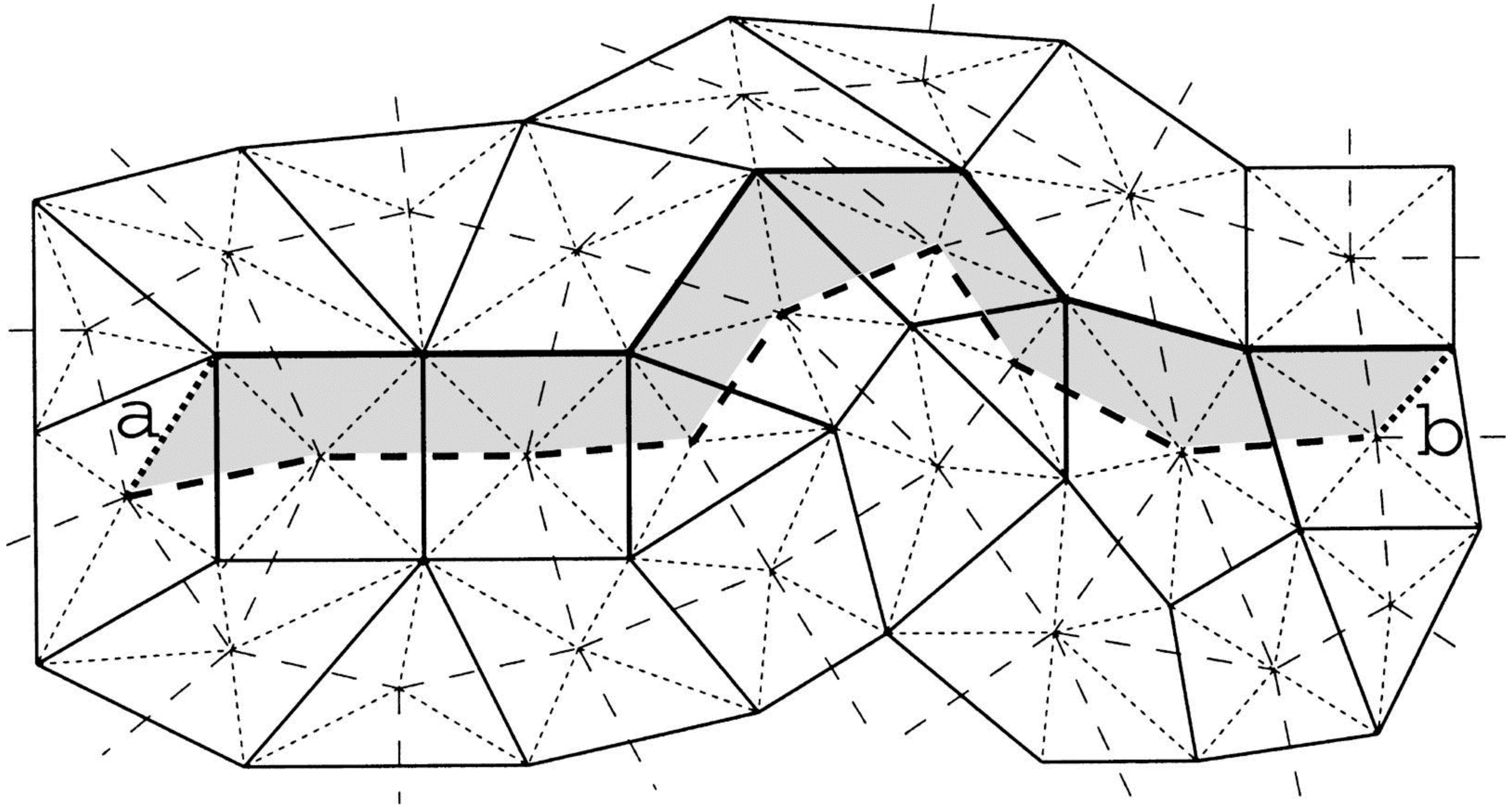
$$A^h(s)B^\varphi(s) = \sum_{(h)} B^{\varphi(S^{-1}(h^{(3)})\bullet h^{(1)})}(s)A^{h^{(2)}}(s),$$

Hamiltonian

Set h and φ as Haar integrals of H and $\hat{H}$

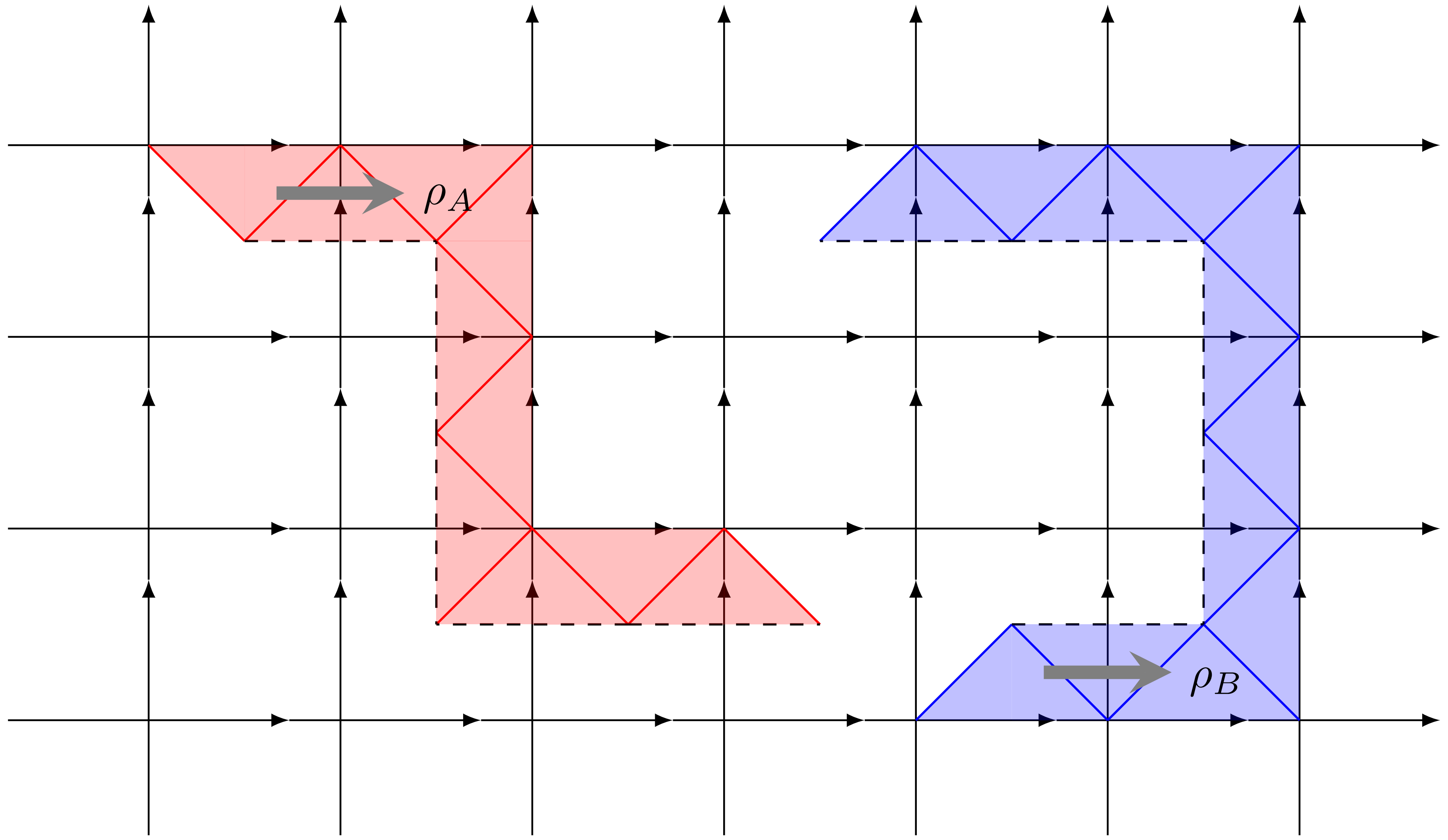
$$H = \sum_v (I - A_v) + \sum_f (I - B_f)$$

[Buerschaper, Mombelli, Christandl, Aguado, 2013]



Ribbon operator and topological excitation

Two types of ribbon



type-A: $D(H)^{\vee,\mathrm{op}}$
type-B: $D(H)^{\vee}$

[Jia, Kaszlikowski, Tan, 2023]

Example: type-B $D(H)^{\vee}$

Dual Hopf algebra $D(H)^{\vee} = D_B(H)^{\vee} = H^{\mathrm{op}} \otimes \hat{H}$, with

$$(h \otimes \varphi)(g \otimes \psi) = gh \otimes \varphi\psi,$$

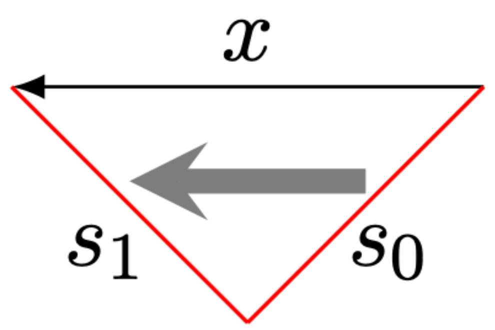
$$\Delta_{D(H)^{\vee}}(h \otimes \alpha) = \sum_{k,(k),(h)} (h^{(1)} \otimes \hat{k}) \otimes (S(k^{(3)})h^{(2)}k^{(1)} \otimes \alpha(k^{(2)}\bullet)),$$

$$1_{D(H)^{\vee}} = 1_H \otimes \varepsilon_H, \quad \varepsilon_{D(H)^{\vee}}(h \otimes \alpha) = \varepsilon(h)\alpha(1),$$

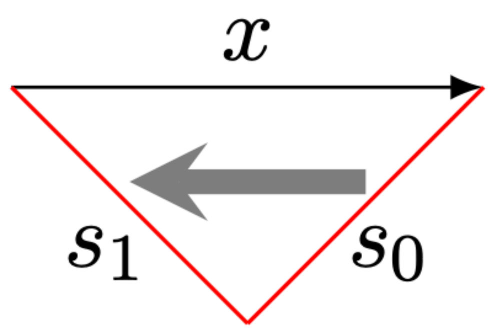
$$\begin{aligned} F^{h,\varphi}(\rho) &= \sum_{(h\otimes\varphi)} F^{(h\otimes\varphi)^{(1)}}(\tau_1) F^{(h\otimes\varphi)^{(2)}}(\tau_2) \\ &= \sum_{k} \sum_{(k),(h)} F^{h^{(1)},\hat{k}}(\tau_1) F^{S(k^{(3)})h^{(2)}k^{(1)},\varphi(k^{(2)}\bullet)}(\tau_2). \end{aligned}$$

Ribbon operator and topological excitation

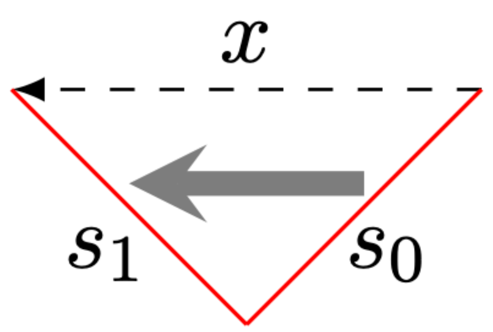
Triangle operators



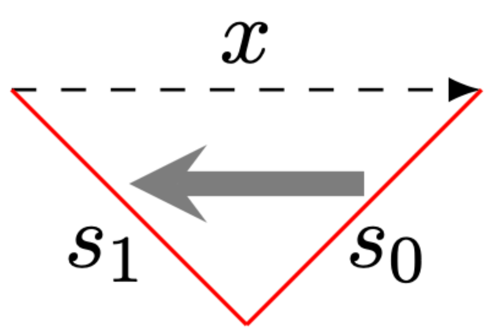
$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T^{\varphi}_{-}|x\rangle = \varepsilon(h)|x \leftharpoonup \hat{S}(\varphi)\rangle,$$



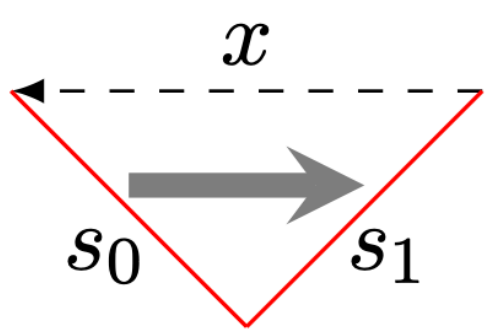
$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T^{\varphi}_{+}|x\rangle = \varepsilon(h)|\varphi \rightharpoonup x\rangle.$$



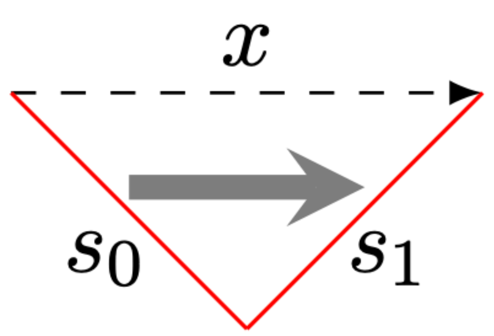
$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L^{h}_{-}|x\rangle = \hat{\varepsilon}(\varphi)|x \triangleleft S(h)\rangle,$$



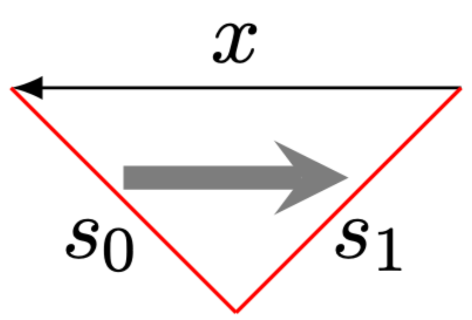
$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L^{h}_{+}|x\rangle = \hat{\varepsilon}(\varphi)|h \triangleright x\rangle.$$



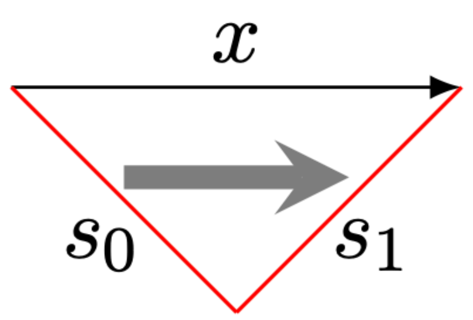
$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}^{h}_{-}|x\rangle = \hat{\varepsilon}(\varphi)|x \triangleleft h\rangle.$$



$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_{+}|x\rangle = \hat{\varepsilon}(\varphi)|S(h) \triangleright x\rangle.$$



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}^{\varphi}_{-}|x\rangle = \varepsilon(h)|x \leftharpoonup \varphi\rangle,$$



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}^{\varphi}_{+}|x\rangle = \varepsilon(h)|\hat{S}(\varphi) \rightharpoonup x\rangle.$$

[Jia, Kaszlikowski, Tan, 2023]

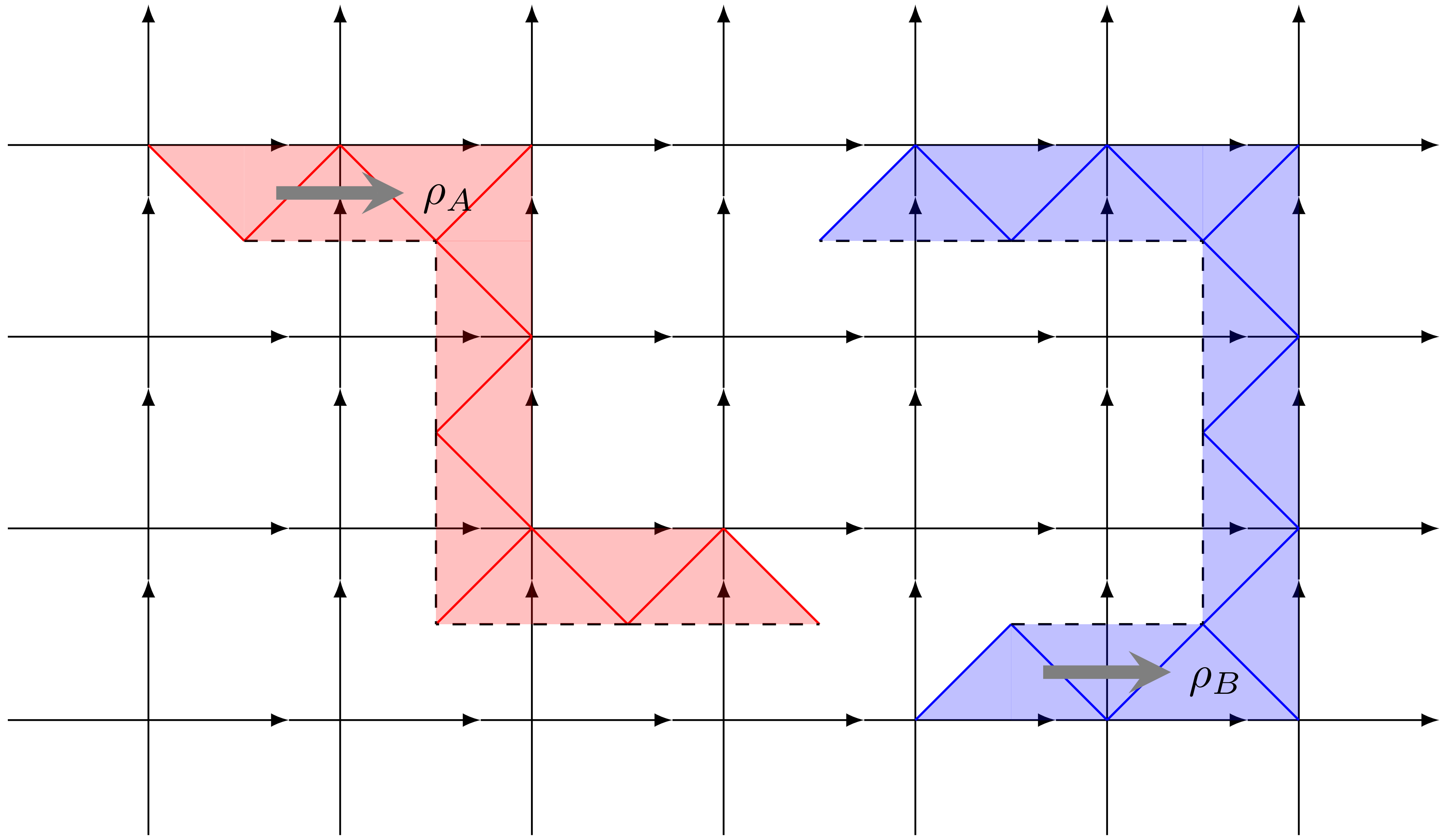
Ribbon operator and topological excitation

Topological excitations: finite group

A topological charge is an irrep of D(H)

$$a_{[g],\pi} = \mathbb{C}[G] \otimes_{\mathbb{C}C_G(g)} M_\pi$$

- Electric charge g=1
- Magnetic charge: trivial rep
- Dyonic charge: electric + magenatic



Hopf quantum double excitation

For general Hopf case D(H)

$$a_{g,M_g} = H \otimes_{H_g} M_g$$

$$\mathrm{FPdim}\, a_{g,M_g} = \frac{|G|}{\dim H_g} \dim M_g$$

- There is a universal grading group $G = U(\mathsf{Rep}(H))$ for any semisimple Hopf algebra H.
- Consider the largest central Hopf subalgebra $K(\hat{H}^{\mathrm{cop}})$ of $\hat{H}^{\mathrm{cop}}$, we have $K(\hat{H}^{\mathrm{cop}}) = \mathbb{C}G^\vee$, which is commutative, and $\mathsf{Rep}(\hat{H}^{\mathrm{cop}}) = \oplus_{g\in G}\mathsf{Rep}(\hat{H}^{\mathrm{cop}})_g$. Suppose that H_g is a Hopf subalgebra of H such that $\mathsf{Rep}(\hat{H}_g) = \oplus_{x\in C_G(g)}\mathsf{Rep}(\hat{H}^{\mathrm{cop}})_x$. It is proved that $K(\hat{H}^{\mathrm{cop}}) = \mathbb{C}G^\vee$ is a normal Hopf subalgebra of $D(H)$.
- $\mathcal{I}_g = \{M_g\}$ is the set of all irreducible representations of $\hat{H}^{\mathrm{cop}} \bowtie H_g$ (here "$\bowtie$" denotes bicrossed product) such that the character χ_{M_g}, when restricted on $K(\hat{H}^{\mathrm{cop}})$, satisfies $\chi_{M_g}|_{K(\hat{H}^{\mathrm{cop}})} = g \dim M_g$.

[Jia, Kaszlikowski, Tan, 2023]

Gapped boundary: algebraic theory

	Hopf quantum double model	String-net model
Bulk	Hopf algebra H	UFC $\mathcal{C} = \mathsf{Rep}(H)$
Bulk phase	$\mathcal{D} = \mathsf{Rep}(D(H))$	$\mathsf{Fun}_{\mathsf{Rep}(H)\vert\mathsf{Rep}(H)}(\mathsf{Rep}(H), \mathsf{Rep}(H))$
Boundary	H-comodule algebra $\mathfrak{A}$	$\mathsf{Rep}(H)$-module category ${}_{\mathfrak{A}}\mathcal{M} = {}_{\mathfrak{A}}\mathsf{Mod}$
Boundary phase	$\mathcal{B} \simeq {}^{H}_{\mathfrak{A}}\mathsf{Mod}_{\mathfrak{A}}$	$\mathsf{Fun}_{\mathsf{Rep}(H)}({}_{\mathfrak{A}}\mathcal{M}, {}_{\mathfrak{A}}\mathcal{M})$
Boundary defect	${}^{H}_{\mathfrak{B}}\mathsf{Mod}_{\mathfrak{A}}$	$\mathsf{Fun}_{\mathsf{Rep}(H)}({}_{\mathfrak{A}}\mathcal{M}, {}_{\mathfrak{B}}\mathcal{M})$

	Hopf quantum double model	String-net model
Bulk	Hopf algebra H	UFC $\mathcal{C} = \mathsf{Rep}(H)$
Bulk phase	$\mathcal{D} = \mathsf{Rep}(D(H))$	$\mathsf{Fun}_{\mathsf{Rep}(H)\vert\mathsf{Rep}(H)}(\mathsf{Rep}(H), \mathsf{Rep}(H))$
Boundary	H-module algebra $\mathfrak{M}$	$\mathcal{C}_{\mathfrak{M}}$
Boundary phase	$\mathcal{B} \simeq {}_{\mathfrak{M}}\mathcal{C}_{\mathfrak{M}}$	$\mathsf{Fun}_{\mathcal{C}}(\mathcal{C}_{\mathfrak{M}}, \mathcal{C}_{\mathfrak{M}})$
Boundary defect	${}_{\mathfrak{N}}\mathcal{C}_{\mathfrak{M}}$	$\mathsf{Fun}_{\mathcal{C}}(\mathcal{C}_{\mathfrak{M}}, \mathcal{C}_{\mathfrak{N}})$

[Jia, Kaszlikowski, Tan, 2023]

Gapped boundary: algebraic theory

Definition *Let H be a Hopf algebra and $(\mathfrak{A}, \mu_{\mathfrak{A}}, \eta_{\mathfrak{A}})$ an algebra. If $\mathfrak{A}$ is a left H-comodule with left coaction $\beta_{\mathfrak{A}} : \mathfrak{A} \to H \otimes \mathfrak{A}$ such that $\beta_{\mathfrak{A}}(xy) = \beta_{\mathfrak{A}}(x)\beta_{\mathfrak{A}}(y)$ and $\beta_{\mathfrak{A}}(1_{\mathfrak{A}}) = 1_H \otimes 1_{\mathfrak{A}}$, then $\mathfrak{A}$ is called a left H-comodule algebra. A right H-comodule algebra can be defined similarly.*

Definition *Let H be a Hopf algebra and $\mathfrak{M}$ an algebra. If $\mathfrak{M}$ is a left H-module such that $h \triangleright (xy) = \sum_{(h)} (h^{(1)} \triangleright x)(h^{(2)} \triangleright y)$ and $h \triangleright 1_{\mathfrak{M}} = \varepsilon(h) 1_{\mathfrak{M}}$, then $\mathfrak{M}$ is called a left H-module algebra. A right H-module algebra can be defined similarly.*

[Jia, Kaszlikowski, Tan, 2023]

Gapped boundary: Hamiltonian theory

Definition *A symmetric separability idempotent of an algebra* $\mathfrak{A}$ *is an element* $\lambda = \sum_{\langle\lambda\rangle} \lambda^{\langle 1\rangle} \otimes \lambda^{\langle 2\rangle} \in \mathfrak{A}\otimes\mathfrak{A}$ *that satisfies the following conditions:*

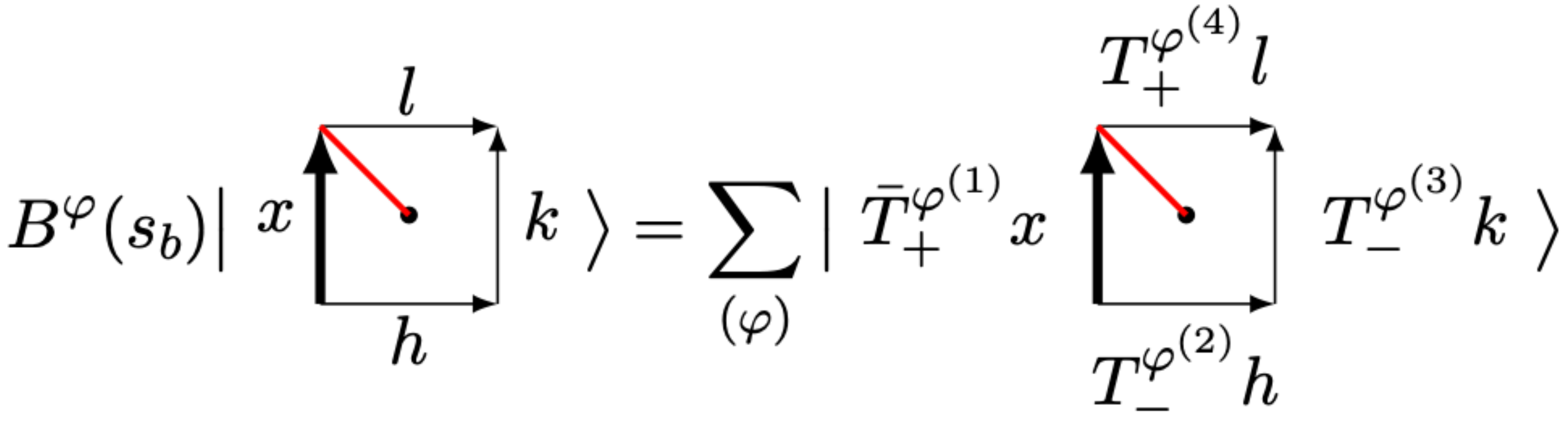
(1) $\sum_{\langle\lambda\rangle} x\lambda^{\langle 1\rangle} \otimes \lambda^{\langle 2\rangle} = \sum_{\langle\lambda\rangle} \lambda^{\langle 1\rangle} \otimes \lambda^{\langle 2\rangle} x$, *for all* $x \in \mathfrak{A}$.

(2) $\sum_{\langle\lambda\rangle} \lambda^{\langle 1\rangle}\lambda^{\langle 2\rangle} = 1.$

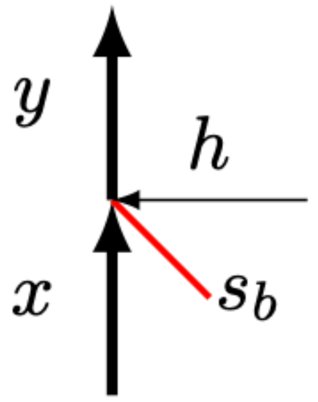
(3) $\sum_{\langle\lambda\rangle} \lambda^{\langle 1\rangle} \otimes \lambda^{\langle 2\rangle} = \sum_{\langle\lambda\rangle} \lambda^{\langle 2\rangle} \otimes \lambda^{\langle 1\rangle}.$

Always exist!

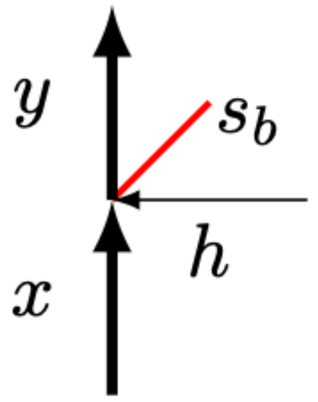
Boundary stabilizers



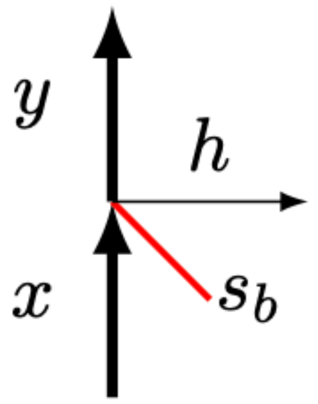
[Jia, Kaszlikowski, Tan, 2023]



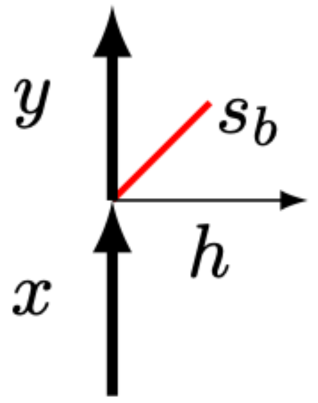
$$A^{a\otimes b}(s_b)|x,y,h\rangle = \sum_{[b]} |ax, yb^{[0]}, S(b^{[1]})h\rangle,$$



$$A^{a\otimes b}(s_b)|x,y,h\rangle = \sum_{[a]} |a^{[0]}x, yb, a^{[1]}h\rangle,$$



$$A^{a\otimes b}(s_b)|x,y,h\rangle = \sum_{[b]} |ax, yb^{[0]}, hb^{[1]}\rangle,$$



$$A^{a\otimes b}(s_b)|x,y,h\rangle = \sum_{[a]} |a^{[0]}x, yb, hS(a^{[1]})\rangle.$$

Gapped boundary: Hamiltonian theory

We introduce the crossed product $(\mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}) \star \hat{H}$ between $\mathfrak{A} \otimes \mathfrak{A}^{\mathrm{op}}$ and $\hat{H}$ as follows: the underlying vector space is $(\mathfrak{A} \otimes \mathfrak{A}) \otimes \hat{H}$, the multiplication is given by

$$((a \otimes b) \star \varphi) \cdot ((c \otimes d) \star \psi) = \sum (ac^{[0]} \otimes d^{[0]}b) \star \varphi(c^{[1]} \bullet d^{[1]})\psi,$$

and the unit is $(1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}}) \star \varepsilon$. Clearly, $((1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}}) \star \varepsilon) \cdot ((a \otimes b) \star \varphi) = ((a \otimes b) \star \varphi) \cdot ((1_{\mathfrak{A}} \otimes 1_{\mathfrak{A}}) \star \varepsilon) = (a \otimes b) \star \varphi$. The multiplication is associative:

$$\begin{aligned}
&[((a \otimes b) \star \varphi) \cdot ((c \otimes d) \star \psi)] \cdot ((e \otimes f) \star \theta) \\
&= \sum ((ac^{[0]} \otimes d^{[0]}b) \star \varphi(c^{[1]} \bullet d^{[1]})\psi) \cdot ((e \otimes f) \star \theta) \\
&= \sum (ac^{[0]}e^{[0]} \otimes f^{[0]}d^{[0]}b) \star \varphi(c^{[1]}e^{[1]} \bullet f^{[1]}d^{[1]})\psi(e^{[2]} \bullet f^{[2]})\theta \\
&= \sum ((a \otimes b) \star \varphi) \cdot ((ce^{[0]} \otimes f^{[0]}d) \star \psi(e^{[1]} \bullet f^{[1]})\theta) \\
&= ((a \otimes b) \star \varphi) \cdot [((c \otimes d) \star \psi) \cdot ((e \otimes f) \star \theta)].
\end{aligned}$$

[Jia, Kaszlikowski, Tan, 2023]

Straightening relation: $\varphi(a \otimes b) = \sum (a^{[0]} \otimes b^{[0]})\varphi(a^{[1]} \bullet b^{[1]})$

Gapped boundary: Hamiltonian theory

Proposition *At a boundary site s_b, the boundary face and vertex operators generate the algebra $(\mathfrak{A}\otimes\mathfrak{A}^{\mathrm{op}})\star\hat{H}$, with the straightening relation*

$$B^{\varphi}(s_b)A^{a\otimes b}(s_b)=\sum_{[a],[b]}A^{a^{[0]}\otimes b^{[0]}}(s_b)B^{\varphi(a^{[1]}\bullet b^{[1]})}(s_b),$$

where $a\otimes b\in\mathfrak{A}\otimes\mathfrak{A}^{\mathrm{op}}$ and $\varphi\in\hat{H}$. Therefore, the map

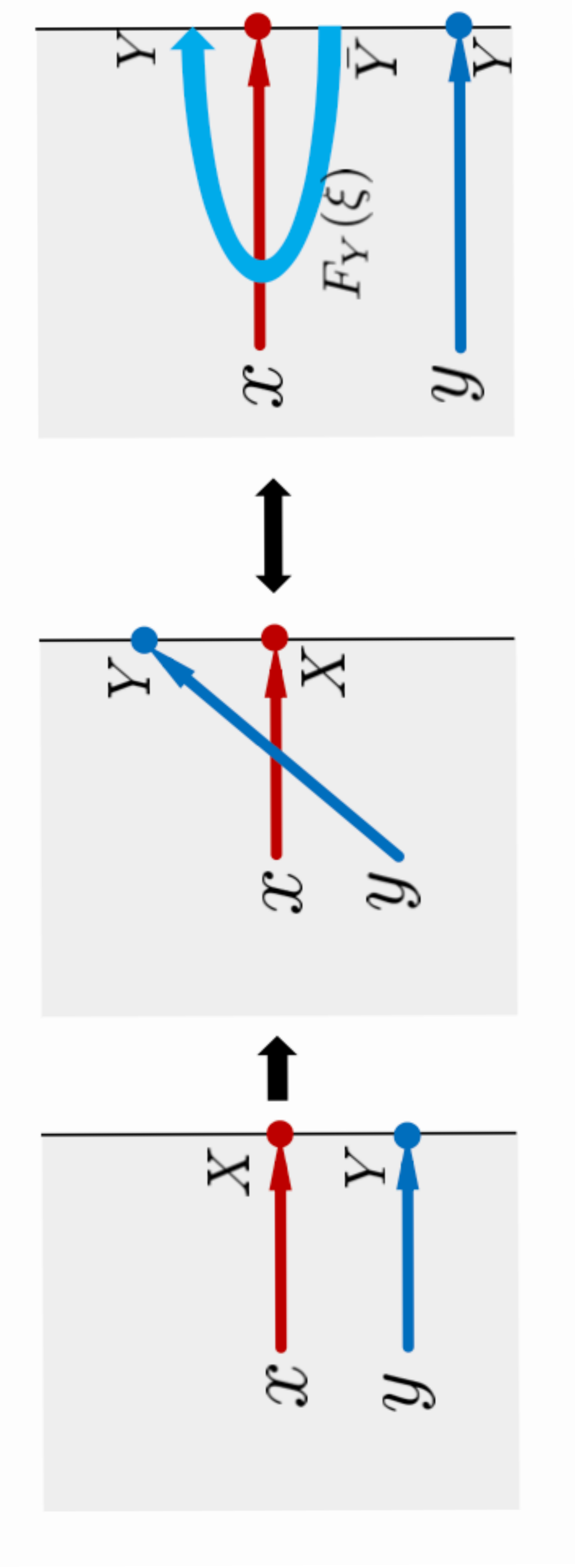
$$\Psi_{s_b}:(\mathfrak{A}\otimes\mathfrak{A}^{\mathrm{op}})\star\hat{H}\to\mathrm{End}(\mathcal{H}(s_b)),\quad (a\otimes b)\star\varphi\mapsto A^{a\otimes b}(s_b)B^{\varphi}(s_b)$$

is an algebra homomorphism. That is, every boundary site supports a representation of $(\mathfrak{A}\otimes\mathfrak{A}^{\mathrm{op}})\star\hat{H}$.

Proposition *Consider a gapped boundary determined by an H-comodule algebra $\mathfrak{A}$:*

1. *The boundary local operator algebra is fusion-categorical Morita equivalent to $\mathcal{D}(s_b)\simeq(\mathfrak{A}\otimes\mathfrak{A}^{\mathrm{op}})\star\hat{H}$.*
2. *The bulk-to-boundary ribbon operator algebra $\mathcal{A}_{\rho_\downarrow}$ is the dual of the boundary local operator algebra $\mathcal{A}_{\rho_\downarrow}\simeq\mathcal{D}(s_b)^{\vee}$. Or equivalently, the dual of the bulk-to-boundary ribbon operator algebra is fusion-categorical Morita equivalent to the boundary local operator algebra $\mathcal{A}_{\rho_\downarrow}^{\vee}\simeq(\mathfrak{A}\otimes\mathfrak{A}^{\mathrm{op}})\star\hat{H}$.*

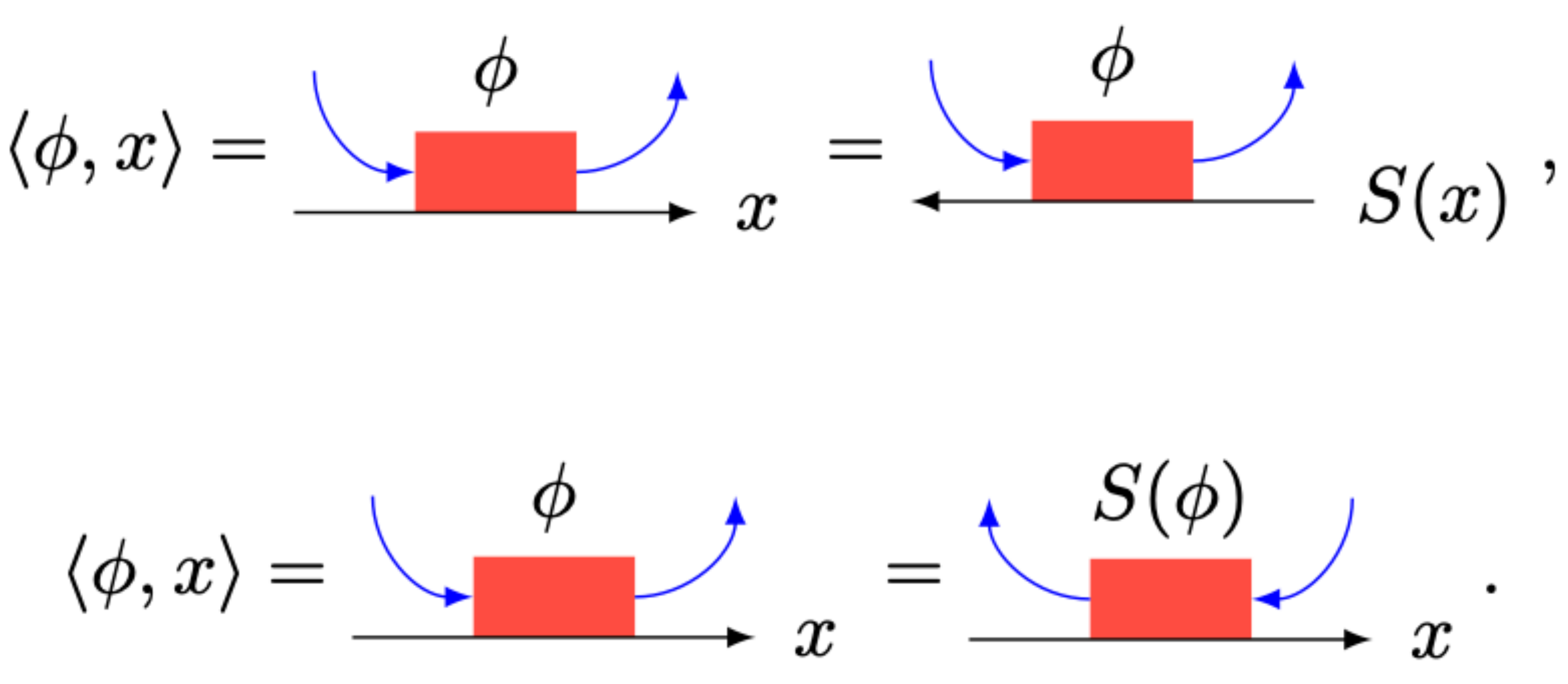
[Jia, Kaszlikowski, Tan, 2023]

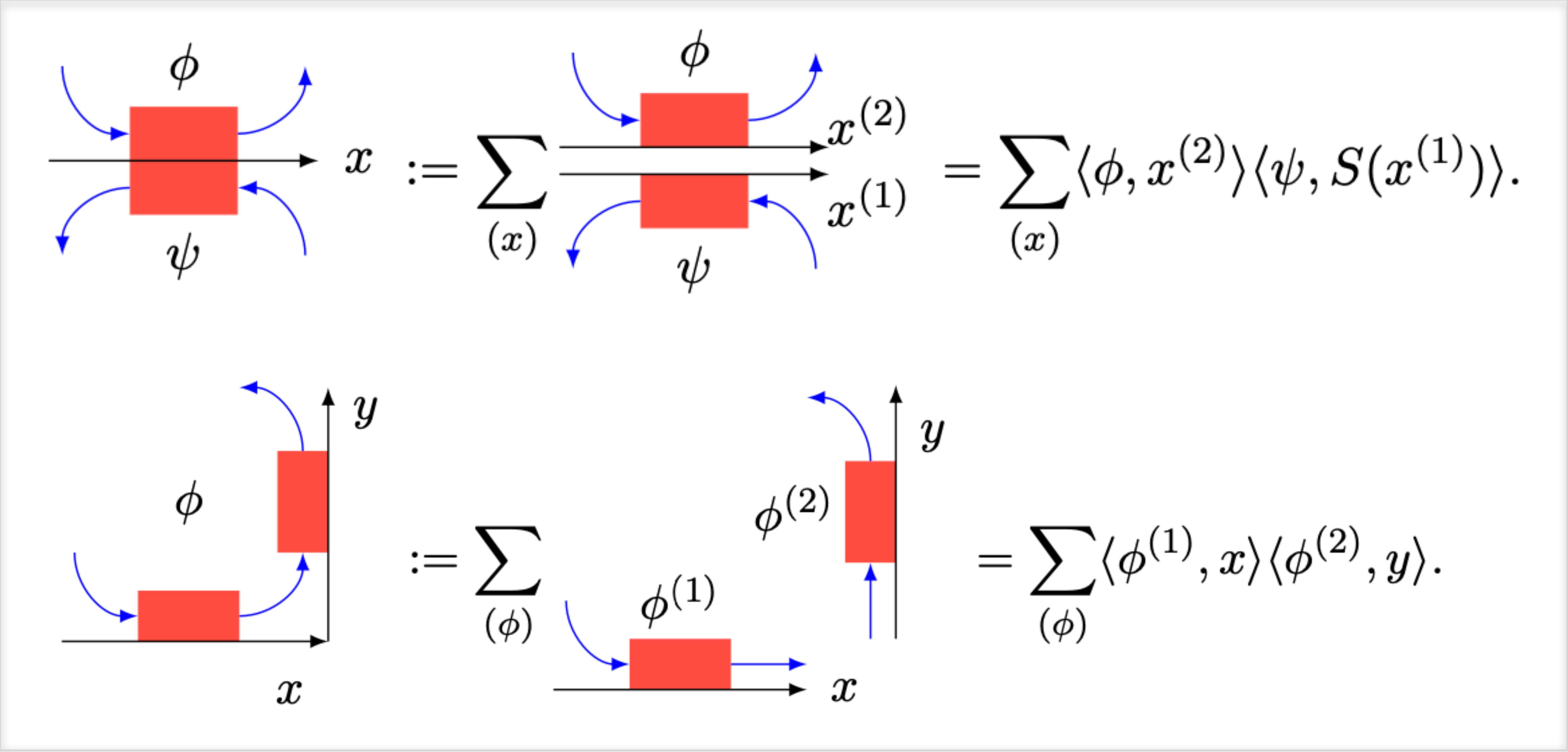


Boudary-bulk duality
Holographic principle

Gapped boundary: Solve the boundary model

Hopf Tensor Network



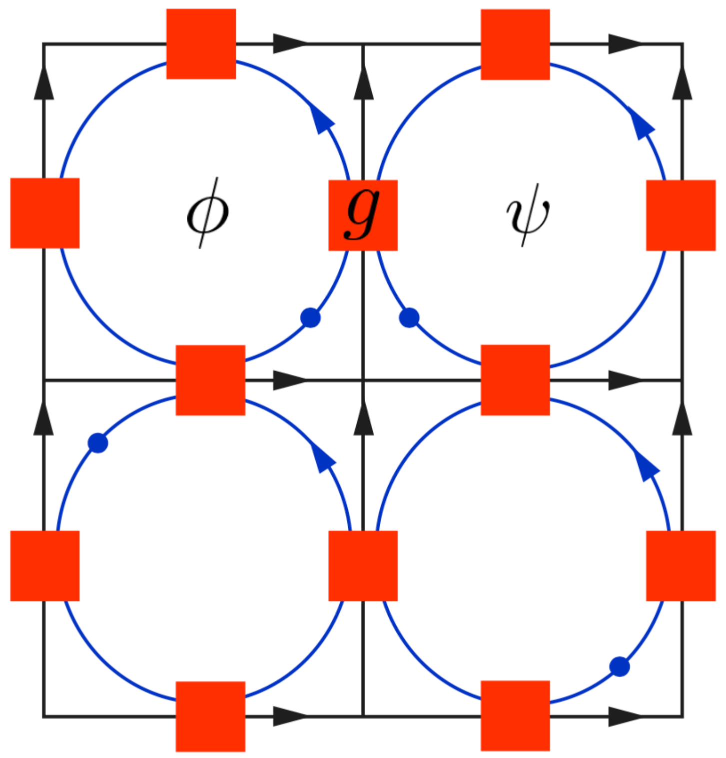


Proposition (Ground state on a disk). *Suppose that Σ is a disk, then the ground state is unique. Then the ground state of the boundary lattice model determined by an H-comodule algebra $\mathfrak{A}$ is the Hopf tensor network state*

$$|\Psi_{GS}\rangle = |\Psi_{C(\Sigma\backslash\partial\Sigma);C(\partial\Sigma)}(\{h_{H,e}\},\{\phi_{\hat{H},f}\};\{h_{\mathfrak{A},e_b}\},\{\varphi_{\hat{H},f_b}\})\rangle.$$

For each bulk site s, we have

$$B^{\varphi}(s)A^{g}(s)|\Psi_{GS}\rangle = \varepsilon(g)\varphi(1_H)|\Psi_{GS}\rangle.$$

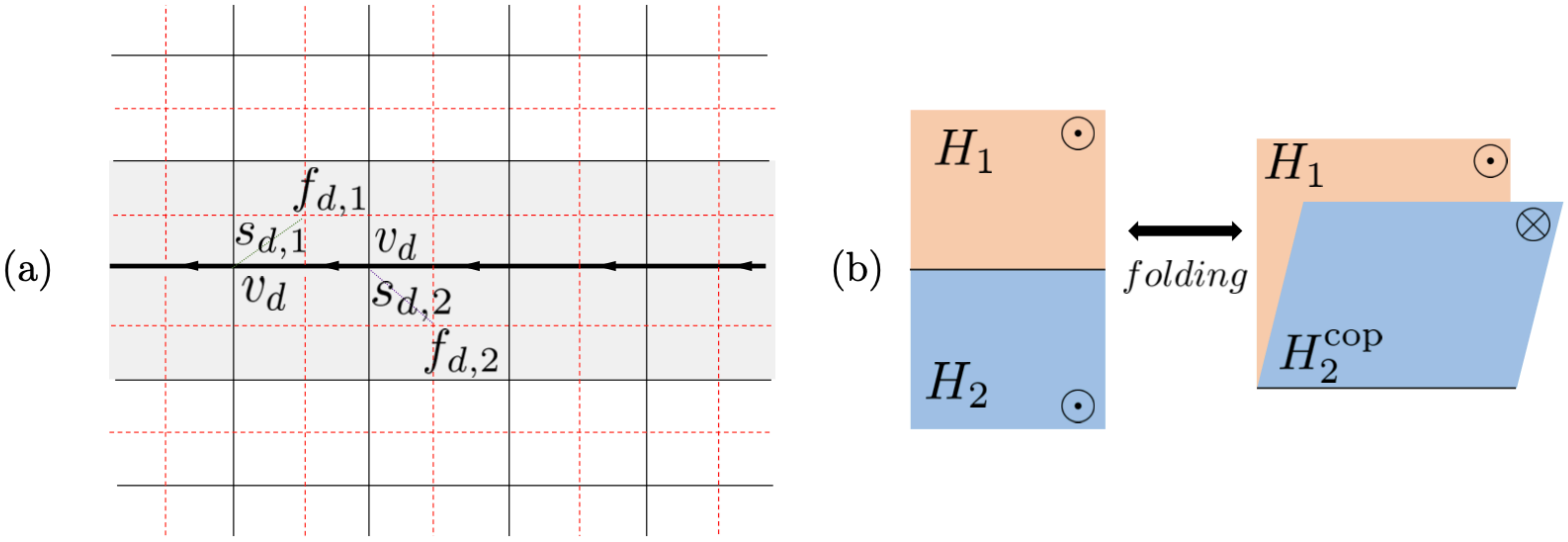


[Jia, Kaszlikowski, Tan, 2023]

Domain Wall: Algebraic Theory

Folding Trick

- Domain wall is a boundary
- Boundary is a domain wall

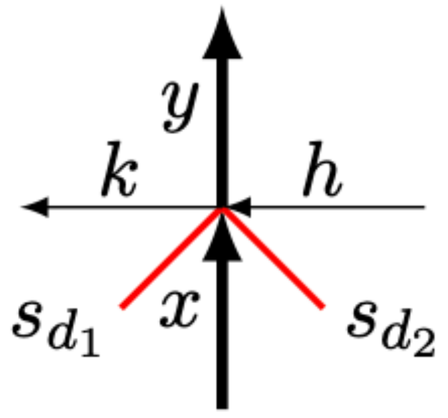


	Hopf quantum double model	String-net model
Bulks ($i=1,2$)	Hopf algebras H_i	UFC $\mathcal{C}_i = \mathsf{Rep}(H_i)$
Bulk phases	$\mathcal{D}_i = \mathsf{Rep}(D(H_i))$	$\mathsf{Fun}_{\mathsf{Rep}(H_i)\|\mathsf{Rep}(H_i)}(\mathsf{Rep}(H_i), \mathsf{Rep}(H_i))$
Domain wall	$H_1\|H_2$-bicomodule algebra $\mathfrak{B}$	$\mathcal{C}_1\|\mathcal{C}_2$-bimodule category ${}_{\mathfrak{B}}\mathsf{Mod}$
Domain wall phase	${}_{\mathfrak{B}}\mathsf{Mod}_{\mathfrak{B}}$	$\mathsf{Fun}_{\mathcal{C}_1\|\mathcal{C}_2}({}_{\mathfrak{B}}\mathcal{M}, {}_{\mathfrak{B}}\mathcal{M})$
Domain wall defect	${}_{\mathfrak{B}}\mathsf{Mod}_{\mathfrak{A}}$	$\mathsf{Fun}_{\mathsf{Rep}(H_1)\|\mathsf{Rep}(H_2)}({}_{\mathfrak{A}}\mathcal{M}, {}_{\mathfrak{B}}\mathcal{M})$

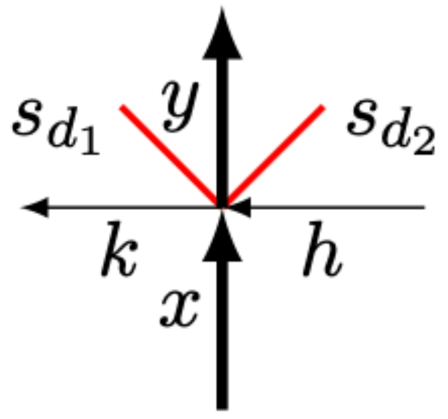
[Jia, Kaszlikowski, Tan, 2023]

Domain Wall: Hamiltonian Theory

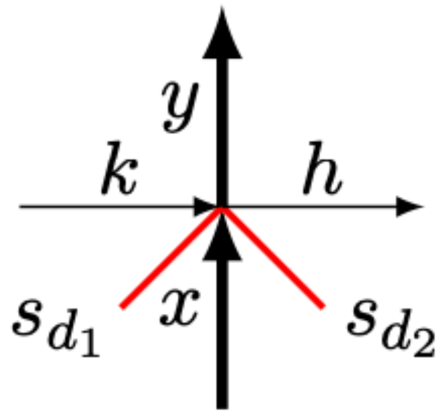
Domain wall lattice



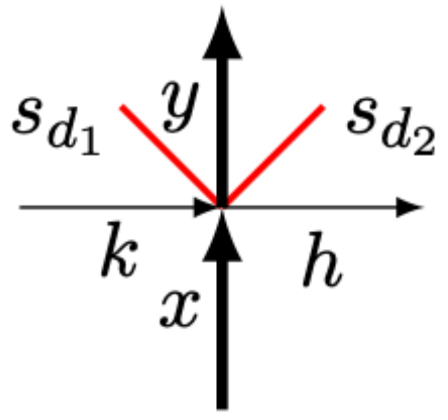
$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[b]} |ax, kb^{[-1]}, yb^{[0]}, S(b^{[1]})h\rangle,$$



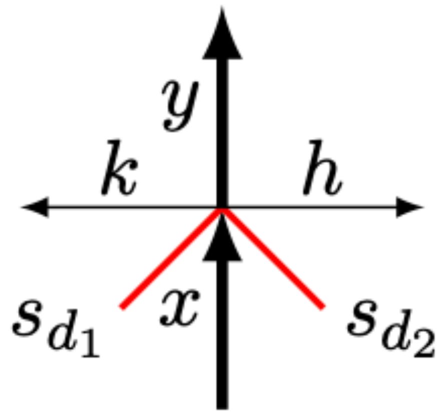
$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[a]} |a^{[0]}x, kS(a^{[-1]}), yb, a^{[1]}h\rangle,$$



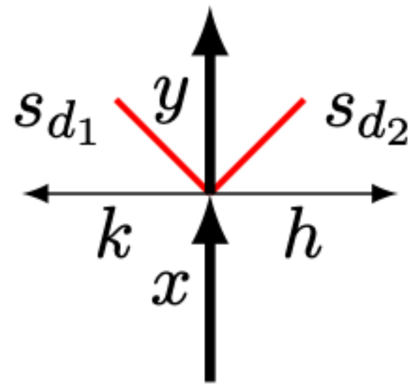
$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[b]} |ax, S(b^{[-1]})k, yb^{[0]}, hb^{[1]}\rangle,$$



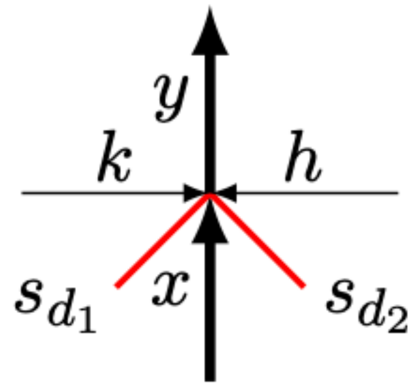
$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[a]} |a^{[0]}x, a^{[-1]}k, yb, hS(a^{[1]})\rangle,$$



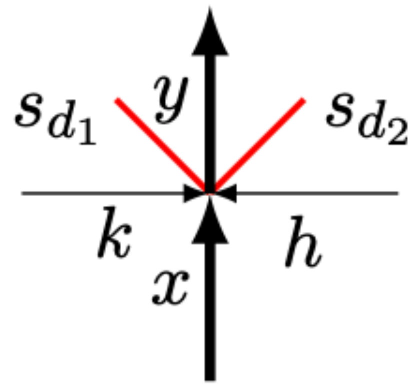
$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[b]} |ax, kb^{[-1]}, yb^{[0]}, hb^{[1]}\rangle,$$



$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[a]} |a^{[0]}x, kS(a^{[-1]}), yb, hS(a^{[1]})\rangle,$$



$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[b]} |ax, S(b^{[-1]})k, yb^{[0]}, S(b^{[1]})h\rangle,$$



$$A^{a\otimes b}(s_{d_i})|x,k,y,h\rangle = \sum_{[a]} |a^{[0]}x, a^{[-1]}k, yb, a^{[1]}h\rangle.$$

$$H[C(\Sigma_d)] = \sum_{i=1,2}\sum_{d_i}(I - A^{\lambda}(s_{d_i})) + \sum_{i=1,2}\sum_{d_i}(I - B^{\varphi_{\hat{H}_i}}(s_{d_i})).$$

[Jia, Kaszlikowski, Tan, 2023]

Discussion

- A complete theory of boundary and domain wall theory of Hopf quantum double
- Weak Hopf case, weak Hopf symmetry ([Jia, Tan, Kaszlikowski, Chang, CMP 2023])
- Higher dimensional case
- Symmetry-enriched case (largely remains open!)
- Entangle entropy is sensitive to defect and boundary
- Operator algebra perspective: stability, Haag duality, infinite-volume sector, etc.
- Weak Hopf quantum double $\leftrightarrow$ extended string-net model

Thank you for your attention!