

Quantum Group Symmetries and Lattice Realizations of Generalized Symmetries

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- ① Quantum group symmetry
- ② Weak Hopf quantum double model
- ③ Weak Hopf tube algebra
- ④ Weak Hopf cluster state and lattice SymTFT

Quantum group symmetry

Anyons and topological order

Anyons are topological quasiparticles whose statistics transcend those of bosons and fermions.

- Topological charges a, b, \dots .
- Fusion rule $a \otimes b = \sum_c N_{ab}^c c$.
- Braiding $c_{a,b}$.
- Topological spin θ_a .
- Antiparticle \bar{a}, \bar{b}, \dots .

Physical signatures of topological excitations

- Long-range entanglement and non-vanishing topological entanglement entropy.
- Topological ground-state degeneracy.
- Anyonic mutual exchange statistics.
- Topological order parameter and topological responses.
- etc.

Anyons and topological order

The mathematical theory of anyons (2d)

- Topological order (TO) is characterized by a unitary modular tensor category (UMTC) \mathcal{C} and a chiral central charge c_- .
- SPT and SET are characterized by (i) modular extension of UMTC or (ii) G -crossed modular tensor category.

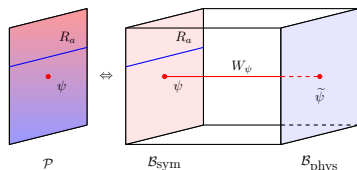
	No TO	TO
No Sym	T. Trivial	TO
Sym	SPT	SET

- Anyon condensation theory and boundary-bulk duality.
- Some notions: gapped/gapless, chiral/non-chiral, anomalous/anomaly-free.
- Generalizations: fermionic TO; mixed state TO; generalized symmetry and their SPT/SET; higher dim.

Generalized symmetry and SymTFT/SymTO

Topological holography or SymTFT/SymTO:

Roughly speaking: a correspondence between d -dimensional symmetry category \mathcal{C} and $d+1$ -dimensional topological order $\mathcal{Z}(\mathcal{C})$.

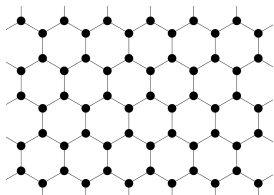


- **Ordinary symmetry:** 0-form symmetry acting on the whole space, obey group law.
- **Non-invertible symmetry:** Topological operators that do not form a group [In this talk: [Our work on categorical symmetry and \(weak\) Hopf symmetry](#), [JHEP 2023,2024a,b, 2025a,b, 2026](#), [CMP 2023](#), with L. Chang, D. Kaszlikowski, S. Tan.].
- Other types: higher form symmetry, modulated symmetry (subsystem, dipolar, fractal), strong/weak symmetry, etc.

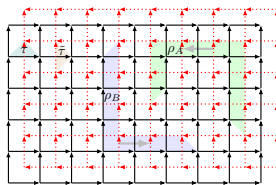
Lattice model of 2d topological order

In this talk, I will mainly discuss two classes of lattice models:

- Kitaev quantum double (QD) model. The original Kitaev QD model is based on a finite group. [Kitaev(1998)].
- Levin-Wen string-net (SN) model. The original Levin-Wen SN model is based on a unitary fusion category (UFC). [Levin, Wen (2005)].
- SN with input UFC \Leftrightarrow QD with input a connected WHA.
- Multifusion SN \Leftrightarrow QD with input a general WHA.
- The symmetry behind 2d topological phase is **weak Hopf symmetry**.



String-net (SN) model

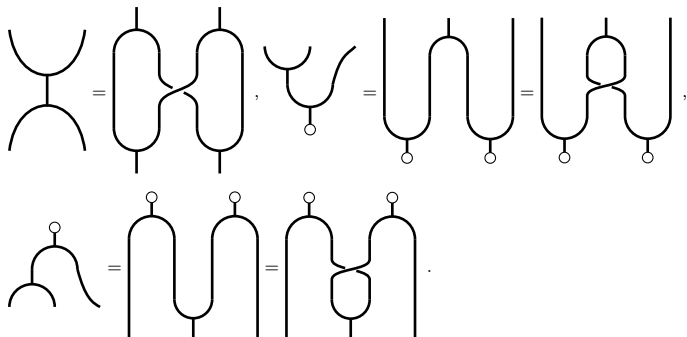


Quantum double (QD) model

Weak Hopf symmetry

Weak Bialgebra

In a braided fusion category \mathcal{C} , a WBA is an object $W \in \mathcal{C}$ that equipped with the following structure (i) Algebra (W, μ, η) (ii) Coalgebra (W, μ, ε) , such that



Weak Hopf symmetry

Weak Hopf algebra

A WHA in \mathcal{C} is a WBA equipped with an antipode $S : W \rightarrow W$ such that

$$\begin{array}{c} \text{Diagram 1: } \boxed{S} \text{ on a loop with top circle} \\ \text{Diagram 2: } \boxed{S} \text{ on a loop with bottom circle} \\ \text{Diagram 3: } \boxed{S} \text{ on a vertical line} \end{array} = \begin{array}{c} \text{Diagram 4: } \boxed{S} \text{ on a loop with top circle} \\ \text{Diagram 5: } \boxed{S} \text{ on a loop with bottom circle} \\ \text{Diagram 6: } \boxed{S} \text{ on a loop with top circle and } \boxed{S} \text{ on a loop with bottom circle} \end{array} .$$

Take $\mathcal{C} = \text{Vect}_{\mathbb{C}}$, we obtain the complex WHA.

- $\varepsilon_L(h) = (\varepsilon \otimes \text{id})(\Delta(1_W)(h \otimes 1_W)) = \sum_{(1_W)} \varepsilon(1_W^{(1)} h) 1_W^{(2)}$ and we denote $W_L = \varepsilon_L(W)$;
- $\varepsilon_R(h) = (\text{id} \otimes \varepsilon)((1_W \otimes h)\Delta(1_W)) = \sum_{(1_W)} 1_W^{(1)} \varepsilon(h 1_W^{(2)})$ and we denote $W_R = \varepsilon_R(W)$.
- A WHA with $S^2 = \text{id}$ is called a weak Kac Algebra.

Weak Hopf symmetry

Definition (Weak Hopf symmetry)

A Hamiltonian $\mathbb{H} : \mathcal{H} \rightarrow \mathcal{H}$ is said to possess weak Hopf symmetry H if \mathcal{H} supports a representation $W_\bullet : H \rightarrow \text{End}(\mathcal{H})$ of the weak Hopf algebra H , such that $[W_h, \mathbb{H}] = 0$ for all $h \in H$. In some contexts, we consider symmetries that act only on the ground states, where $[\mathbb{H}, W_h] = 0$ holds (at least) within the ground state space \mathcal{V}_{GS} . In such cases, the symmetry of the ground state space must be larger than that of the Hamiltonian, i.e., $\text{Sym}_{\text{GS}} \supseteq \text{Sym}_{\mathbb{H}}$. If the ground state space is non-degenerate, meaning $\dim \mathcal{V}_{\text{GS}} = 1$, the ground state must be an eigenstate of the symmetry operator W_h . When the weak Hopf algebra is a weak Kac algebra, we refer to the corresponding symmetry as weak Kac symmetry.

Quantum double model has weak Hopf symmetry (any non-chiral topological order has weak Hopf symmetry!).

WHA symmetries are ubiquitous in 2d TO:

- Hopf symmetry is a special type of weak Hopf symmetry.
- Group symmetry is always a Hopf symmetry, $H = \mathbb{C}[G]$, thus also a special case of WHA.
- Charge symmetry and gauge symmetry. Charges are irreps of charge symmetry. In 2d non-chiral phase: Charge symmetry = Drinfeld double of gauge symmetry.
- Given fusion category \mathcal{C} there always exists a WHA such that $\mathcal{C} \simeq \text{Rep}(W)$.
 - anomaly-free = Hopf algebra; anomalous=weak Hopf algebra.
 - How to recover WHA from given fusion category:
 - (1) Tannaka-Krein reconstruction;
 - (2) **Tube algebra.**

Weak Hopf quantum double model

Drinfeld quantum double of WHA

Drinfeld quantum double

Quantum double $D(W) = (\hat{W} \otimes W)/J$ is equipped WHA structure:

(1) The multiplication

$$[\varphi \otimes h][\psi \otimes g] = \sum_{(\psi), (h)} [\varphi\psi^{(2)} \otimes h^{(2)}g] \langle \psi^{(1)}, S^{-1}(h^{(3)}) \rangle \langle \psi^{(3)}, h^{(1)} \rangle.$$

(2) The unit $[\varepsilon \otimes 1_W]$.

(3) The comultiplication $\Delta([\varphi \otimes h]) = \sum_{(\varphi), (h)} [\varphi^{(2)} \otimes h^{(1)}] \otimes [\varphi^{(1)} \otimes h^{(2)}]$.

(4) The counit $\varepsilon([\varphi \otimes h]) = \langle \varphi, \varepsilon_R(S^{-1}(h)) \rangle$.

(5) The antipode

$$S([\varphi \otimes h]) = \sum_{(\varphi), (h)} [\hat{S}^{-1}(\varphi^{(2)}) \otimes S(h^{(2)})] \langle \varphi^{(1)}, h^{(3)} \rangle \langle \varphi^{(3)}, S^{-1}(h^{(1)}) \rangle.$$

The linear span J of the elements

$$\begin{aligned} \varphi \otimes xh - \varphi(x \rightarrow \varepsilon) \otimes h, & \quad x \in W_L, \\ \varphi \otimes yh - \varphi(\varepsilon \leftarrow y) \otimes h, & \quad y \in W_R, \end{aligned}$$

is a two-sided ideal of $\hat{W}^{\text{cop}} \otimes W$. We denote the quotient algebra $(\hat{W}^{\text{cop}} \otimes W)/J$ as $D(W)$ and equivalent classes in $D(W)$ as $[\varphi \otimes h]$ for $\varphi \otimes h \in \hat{W}^{\text{cop}} \otimes W$. The representation category of $D(W)$ is braided.

Generalities of QD model:

- For an oriented surface, make cellulation $C(\Sigma)$, construct a lattice model on this lattice and show it is topological.
- Local space is chosen as a WHA and put on edges.
- Construct vertex and face operator such that they form a representation of quantum double $D(W)$ on local site.
- Construct ribbon operators which generate topological excitations at its ends.

Lattice model of 2d topological order

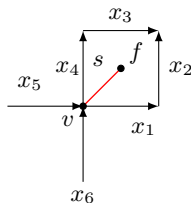
For weak Hopf quantum double model $D(W)$:

- W -action: $L_+^g|z\rangle = |gz\rangle$, $L_-^g|z\rangle = |zS^{-1}(g)\rangle$.
- \hat{W} -action: $T_+^\varphi|x\rangle = |\varphi \rightarrow x\rangle = |\sum_{(x)} \langle \varphi, x^{(2)} \rangle x^{(1)}\rangle$,
 $T_-^\varphi|x\rangle = |x \leftarrow \hat{S}(\varphi)\rangle = |\sum_{(x)} \langle \hat{S}(\varphi), x^{(1)} \rangle x^{(2)}\rangle$.
- Local operators:

$$A^h(s) = \sum_{(h)} L_-^{h(1)}(j_4) \otimes L_+^{h(2)}(j_5) \otimes L_+^{h(3)}(j_6) \otimes L_-^{h(4)}(j_1)$$

$$B^\varphi(s) = \sum_{(\varphi)} T_-^{\varphi(1)}(j_1) \otimes T_-^{\varphi(2)}(j_2) \otimes T_+^{\varphi(3)}(j_3) \otimes T_+^{\varphi(4)}(j_4)$$

$$A^h(s)B^\varphi(s) = \sum_{(h)} B^{\varphi(S^{-1}(h^{(3)}) \bullet h^{(1)})}(s) A^{h^{(2)}}(s)$$



This forms a representation of $D(W)$.

Lattice model of 2d topological order

For weak Hopf quantum double model $D(W)$:

- Set h as Haar integral for vertex operator
 $A_v = A_v^h$.
- Set φ as Haar measure for face operator
 $B_f = B_f^\varphi$.
- The Hamiltonian is given by

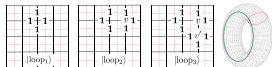
$$H = - \sum_v A_v - \sum_f B_f$$

- The model has WHA symmetry.
- Typical example is toric code

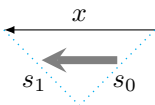
$$H = - \sum_v XXXX - \sum_f ZZZZ$$

Haar integral

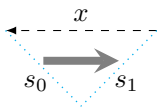
A left (resp. right) integral of W is an element l (resp. r) satisfying $xl = \varepsilon_L(x)l$ (resp. $rx = r\varepsilon_R(x)$). A left (resp. right) integral l (resp. r) is called left (resp. right) normalized if $\varepsilon_L(l) = 1_W$ (resp. $\varepsilon_R(r) = 1_W$). If h is both a left and right integral, it is called a two-side integral. A Haar integral in W (or Haar measure on \hat{W}) is a two-side normalized two-side integral.



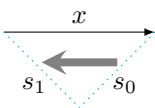
Triangle operators



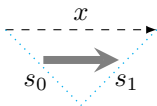
$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T_-^\varphi|x\rangle \\ = \varepsilon(h)|x \leftarrow \hat{S}(\varphi)\rangle,$$



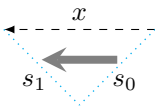
$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_-^h|x\rangle \\ = \hat{\varepsilon}(\varphi)|x \triangleleft h\rangle.$$



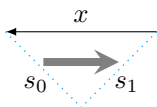
$$F^{h,\varphi}(\tau_R)|x\rangle = \varepsilon(h)T_+^\varphi|x\rangle \\ = \varepsilon(h)|\varphi \rightarrow x\rangle.$$



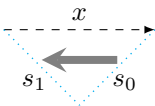
$$F^{h,\varphi}(\tilde{\tau}_L)|x\rangle = \hat{\varepsilon}(\varphi)\tilde{L}_+^h|x\rangle \\ = \hat{\varepsilon}(\varphi)|S(h) \triangleright x\rangle.$$



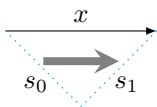
$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L_-^h|x\rangle \\ = \hat{\varepsilon}(\varphi)|x \triangleleft S(h)\rangle,$$



$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}_-^\varphi|x\rangle \\ = \varepsilon(h)|x \leftarrow \varphi\rangle,$$



$$F^{h,\varphi}(\tilde{\tau}_R)|x\rangle = \hat{\varepsilon}(\varphi)L_+^h|x\rangle \\ = \hat{\varepsilon}(\varphi)|h \triangleright x\rangle.$$

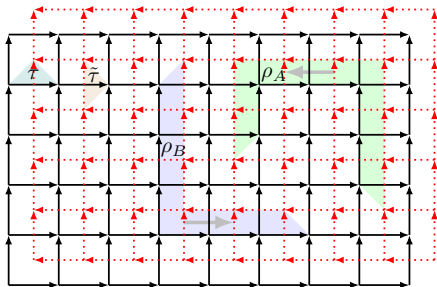


$$F^{h,\varphi}(\tau_L)|x\rangle = \varepsilon(h)\tilde{T}_+^\varphi|x\rangle \\ = \varepsilon(h)|\hat{S}(\varphi) \rightarrow x\rangle.$$

Ribbon operators

Ribbon operators create topological excitations.

$$F^{h,\varphi}(\rho) = \sum_k \sum_{(k),(h)} F^{h^{(1)},\hat{k}}(\rho_1) F^{S^{-1}(k^{(3)})h^{(2)}k^{(1)},\varphi(k^{(2)})\bullet}(\rho_2)$$



The type-B ribbon operators form a dual algebra $D(W)^\vee$ of $D(W)$; the type-A ribbon operators form $D(W)^{\vee, \text{op}}$.

Topological boundary and boundary-bulk duality

- Boundary gauge symmetry is W -comodule algebra, boundary charge symmetry is a WHA.
- Boundary Hamiltonian is constructed via replace Haar integral with symmetric separability idempotent.

A *symmetric separability idempotent* of an algebra A is an element $\Upsilon \in A \otimes A$, expressed in Sweedler notation as $\Upsilon = \sum_{\langle \Upsilon \rangle} \Upsilon^{(1)} \otimes \Upsilon^{(2)} := \sum_i \Upsilon_i^{(1)} \otimes \Upsilon_i^{(2)}$, that satisfies the following three conditions:

- Centrality: for all $x \in A$,

$$\sum_{\langle \Upsilon \rangle} x \Upsilon^{(1)} \otimes \Upsilon^{(2)} = \sum_{\langle \Upsilon \rangle} \Upsilon^{(1)} \otimes \Upsilon^{(2)} x;$$

- Normalization: $\sum_{\langle \Upsilon \rangle} \Upsilon^{(1)} \Upsilon^{(2)} = 1_A$;
- Symmetry: $\sum_{\langle \Upsilon \rangle} \Upsilon^{(1)} \otimes \Upsilon^{(2)} = \sum_{\langle \Upsilon \rangle} \Upsilon^{(2)} \otimes \Upsilon^{(1)}$.

- The boundary phase is a UFC \mathcal{B} , the boundary-bulk duality is given by $\text{Rep}(D(W)) \simeq \mathcal{Z}(\mathcal{B})$.

Anyon condensation as weak Hopf symmetry breaking

Theorem (weak Hopf symmetry breaking) [Jia, Tan, Kaszlikowski, Chang 2023]

Consider a quantum phase $\mathcal{C} = \text{Rep}(W)$ with weak Hopf symmetry W , after the formation of condensate, the symmetry is broken into a weak Hopf subalgebra $V \hookrightarrow W$. The new phase is given by $\mathcal{D} = \text{Rep}(V)$. In this new phase, the particles that have nontrivial monodromy with condensate are confined, and the particles that have trivial monodromy with condensate are deconfined. The set \mathcal{A}_{dc} of deconfined particles are fusion closed. These deconfined particles are irreducible representations of a new weak Hopf symmetry U , which is a weak Hopf quotient of V .

$$\begin{array}{ccccc} & \text{Rep}(W) & \xrightarrow{F_1} & \text{Rep}(V) & \xrightarrow{F_2} & \text{Rep}(U) \\ \text{Symmetry breaking:} & \updownarrow & & \updownarrow & & \updownarrow \\ & W & \xleftarrow{\iota} & V & \xrightarrow{\pi} & U \end{array} \quad (1)$$

where F_1 and F_2 are monoidal functors, ι is an injective weak Hopf map (embedding), and π is a surjective weak Hopf map (quotient).

Weak Hopf tube algebra

String-net model

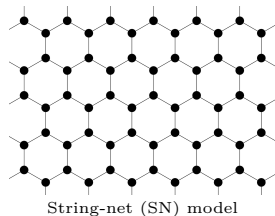
String-net

A connected oriented trivalent lattice Σ is called a string-net. For a string-net model with an input UMFC \mathcal{D} , its topological excitation is given by UMTC $\mathcal{Z}(\mathcal{D})$.

Vertex space \mathcal{H}_v : (gauge choice $Y_c^{ab} = (d_a d_b / d_c)^{1/2}$)

$$(Y_c^{ab})^{-1/2} \begin{array}{c} a \quad b \\ \searrow \quad \nearrow \\ \alpha \\ \uparrow \\ c \end{array} = |c \rightarrow a, b; \alpha\rangle,$$

$$(Y_c^{ab})^{-1/2} \begin{array}{c} c \\ \uparrow \\ \beta \\ \nearrow \quad \searrow \\ a \quad b \end{array} = \langle a, b \rightarrow c; \beta|.$$



Total Hilbert space $\mathcal{H}_{\text{tot}} = \otimes_v \mathcal{H}_v$.

String-net model

String-net lattice model (vertex and face operators)

$$Q_v \left| \begin{array}{c} a \quad b \\ \searrow \quad \nearrow \\ \alpha \\ \nearrow \\ c \end{array} \right\rangle = \delta_{c \rightarrow a,b} \left| \begin{array}{c} a \quad b \\ \searrow \quad \nearrow \\ \alpha \\ \nearrow \\ c \end{array} \right\rangle.$$

$$B_f^k \left| \begin{array}{c} i_2 \\ \uparrow \\ \alpha_2 \\ \nearrow \quad \searrow \\ \alpha_3 \quad \alpha_1 \\ \nearrow \quad \searrow \\ \alpha_4 \quad \alpha_6 \\ \nearrow \quad \searrow \\ \alpha_5 \quad \alpha_6 \\ \nearrow \quad \searrow \\ i_1 \end{array} \right\rangle = \left| \begin{array}{c} i_2 \\ \uparrow \\ \alpha_2 \\ \nearrow \quad \searrow \\ \alpha_3 \quad \alpha_1 \\ \nearrow \quad \searrow \\ \alpha_4 \quad \alpha_6 \\ \nearrow \quad \searrow \\ \alpha_5 \quad \alpha_6 \\ \nearrow \quad \searrow \\ i_1 \end{array} \right\rangle.$$

String-net lattice model

The local stabilizers Q_v and B_f functions are projectors and mutually commute. Consequently, the Hamiltonian of the generalized multifusion string-net ($J_v, J_f > 0$, $B_f = \sum_{k \in \text{Irr}(\mathcal{D})} w_k B_f^k$, $w_k = Y_{\mathbf{1}}^{k^*k} / \sum_{l \in \text{Irr}(\mathcal{D})} d_l^2$):

$$H = -J_v \sum_v Q_v - J_f \sum_f B_f$$

being a local commutative projector (LCP) Hamiltonian, exhibits a gap in the thermodynamic limit.

Topological local move

• loop move:

$$\begin{array}{c} c' \\ \uparrow \\ \beta \\ \circlearrowleft \\ \alpha \\ \uparrow \\ c \end{array}
 \begin{array}{c} a \\ \leftarrow \\ \circlearrowleft \\ \rightarrow \\ b \end{array}
 = \delta_{c,c'} \delta_{\alpha,\beta} Y_c^{ab}
 \begin{array}{c} c \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}
 .$$

• parallel move:

$$\begin{array}{c} a \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}
 \begin{array}{c} b \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}
 = \sum_{c,\alpha} \frac{1}{Y_c^{ab}}
 \begin{array}{c} a \quad b \\ \searrow \quad \swarrow \\ \quad \alpha \\ \uparrow \\ c \\ \downarrow \\ \quad \alpha \\ \swarrow \quad \searrow \\ a \quad b \end{array}
 .$$

Topological local move

- F-move

$$\begin{aligned} & \text{Diagram 1} = \sum_{n, \mu, \nu} [F_l^{ijk}]_{m\alpha\beta}^{n\mu\nu} \text{Diagram 2} \\ & \text{Diagram 3} = \sum_{m, \alpha, \beta} [(F_l^{ijk})^{-1}]_{n\mu\nu}^{m\alpha\beta} \text{Diagram 4} \\ & \text{Diagram 5} = \sum_{n, \mu, \nu} [F_l^l]_{m\alpha\beta}^{n\mu\nu} \text{Diagram 6} \\ & \text{Diagram 7} = \sum_{m, \alpha, \beta} [(F_l^l)^{-1}]_{n\mu\nu}^{m\alpha\beta} \text{Diagram 8} \end{aligned}$$

Topological local move

The loop move, parallel move, and F-move, collectively known as topological local moves, are equivalent to the Pachner moves.

String-net model

Multifusion string-net model

- For input UMFC \mathcal{D} , the topological excitation is given by UMTC $\mathcal{Z}(\mathcal{D})$ (Drinfeld center).
- The bulk gauge symmetry and charge symmetry are weak Hopf algebras.
- The boundary gauge symmetry is a W -comodule algebra, the boundary charge symmetry is a weak Hopf algebra.
- The domain wall gauge symmetry is a $W_1|W_2$ -comodule algebra, the domain wall charge symmetry is a weak Hopf algebra.

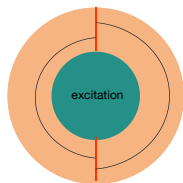
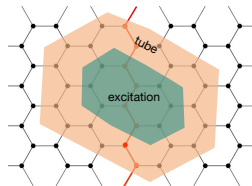
	Weak Hopf QD model	(Multi)fusion SN model
Bulk gauge sym	WHA $\{W_i\}_{i \in I}$	$\{\mathcal{C}_i = \text{Rep}(W_i)\}_{i \in I}$
Bulk phase	$\mathcal{P} = \text{Rep}(D(W_i))$	$\text{Fun}_{\mathcal{C}_i \mathcal{C}_i}(\mathcal{C}_i, \mathcal{C}_i)$
Domain wall gauge symmetry	$\{W_i\}_{i \in I}$ - comultimodule algebra \mathfrak{A}	$\{\mathcal{C}_i\}_{i \in I}$ - multimodule category $\mathcal{M} = {}_{\mathfrak{A}}\text{Mod}$
Domain wall phase	$\mathcal{B} \simeq {}_{\mathfrak{A}}\text{Mod}_{\mathfrak{A}}^{\{W_i\}_{i \in I}}$	$\text{Fun}_{\{\mathcal{C}_i\}_{i \in I}}({}_{\mathfrak{A}}\mathcal{M}, {}_{\mathfrak{A}}\mathcal{M})$
Domain wall defect	$\mathfrak{B} \text{Mod}_{\mathfrak{A}}^{\{W_i\}_{i \in I}}$	$\text{Fun}_{\{\mathcal{C}_i\}_{i \in I}}({}_{\mathfrak{A}}\mathcal{M}, \mathfrak{B}\mathcal{M})$

Weak Hopf tube algebra

Definition (Domain wall tube algebra $\mathbf{Tube}({}_c\mathcal{M}_{\mathcal{D}})$)

For a gapped domain wall ${}_c\mathcal{M}_{\mathcal{D}}$, the domain wall tube algebra $\mathbf{Tube}({}_c\mathcal{M}_{\mathcal{D}})$ is the vector space spanned by the following basis:

$$\left\{ \begin{array}{l} \text{Diagram: A circular diagram with an inner white circle and an outer dashed circle. The region between them is shaded light blue. A red arc labeled 'a' is on the left, and a blue arc labeled 'b' is on the right. A vertical line segment in the center is divided into segments labeled from top to bottom: w, \gamma, v, \zeta, u, z, y, \nu, \mu, x. The segments are connected by arcs: \gamma connects w and v; \zeta connects v and u; u connects u and z; z connects z and y; y connects y and \mu; \mu connects \mu and x. The outer dashed circle is also connected to these segments. The inner white circle is also connected to these segments. The diagram is enclosed in a large left-facing curly bracket. } \quad \begin{array}{l} a \in \text{Irr}(\mathcal{C}), b \in \text{Irr}(\mathcal{D}), x, y, z, u, v, w \in \text{Irr}(\mathcal{M}), \\ \mu \in \text{Hom}_{\mathcal{M}}(x, y \otimes b) \neq 0, \nu \in \text{Hom}_{\mathcal{M}}(y, a \otimes z) \neq 0, \\ \zeta \in \text{Hom}_{\mathcal{M}}(a \otimes u, v) \neq 0, \gamma \in \text{Hom}_{\mathcal{M}}(v \otimes b, w) \neq 0 \end{array} \right\}.$$



Weak Hopf tube algebra

- The unit is given by

$$1 = \sum_{x,z \in \text{Irr}(\mathcal{M})} \text{Diagram} .$$

- The multiplication map μ is defined as

$$\mu \left(\text{Diagram}_1 \otimes \text{Diagram}_2 \right) = \delta_{u,w'} \delta_{z,x'} \text{Diagram}_3 .$$

Weak Hopf tube algebra

- The counit is defined as

$$\varepsilon \left(\begin{array}{c} w \\ \gamma \\ v \mid \zeta \\ u \\ z \\ y \mid \nu \\ \mu \\ x \end{array} \right) = \frac{\delta_{z,u} \delta_{x,w}}{d_x^R} \begin{array}{c} \gamma \\ v \mid \zeta \\ z \\ y \mid \nu \\ \mu \\ \bar{x} \end{array} .$$

- The comultiplication (or coproduct) is defined as

$$\Delta \left(\begin{array}{c} w \\ \gamma \\ v \mid \zeta \\ u \\ z \\ y \mid \nu \\ \mu \\ x \end{array} \right) = \sum_{i,j,k,\rho,\sigma} \sqrt{\frac{d_j^L}{d_a d_i^L}} \sqrt{\frac{d_k^R}{d_j^R d_b}} \begin{array}{c} w \\ \gamma \\ v \mid \zeta \\ u \\ i \\ j \mid \sigma \\ \rho \\ k \end{array} \otimes \begin{array}{c} k \\ \rho \\ j \mid \sigma \\ i \\ z \\ y \mid \nu \\ \mu \\ x \end{array} .$$

Weak Hopf tube algebra

- The antipode is defined as

$$S \left(\begin{array}{c} w \\ \gamma \\ v \\ \zeta \\ u \\ a \quad b \\ z \\ y \\ \nu \\ \mu \\ x \end{array} \right) = \frac{d_u^L d_v^R}{d_v^L d_w^R} \begin{array}{c} z \\ \nu \\ y \\ \mu \\ x \\ \bar{a} \quad \bar{b} \\ w \\ \gamma \\ v \\ \zeta \\ u \end{array} .$$

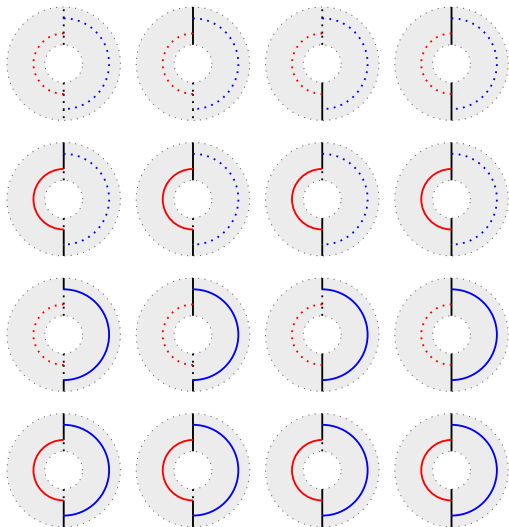
- The $*$ -operation is defined as (set $d^L = d^R$)

$$\left(\begin{array}{c} w \\ \gamma \\ v \\ \zeta \\ u \\ a \quad b \\ z \\ y \\ \nu \\ \mu \\ x \end{array} \right)^* = \frac{d_z}{d_x} \begin{array}{c} u \\ \zeta \\ v \\ \gamma \\ w \\ \bar{a} \quad \bar{b} \\ x \\ \mu \\ y \\ \nu \\ z \end{array} .$$

Theorem

- The domain wall tube algebra $\mathbf{Tube}({}_e\mathcal{M}_{\mathcal{D}})$ is a C^* weak Hopf algebra.
- The representation category $\mathbf{Rep}(\mathbf{Tube}({}_e\mathcal{M}_{\mathcal{D}}))$ of $\mathbf{Tube}({}_e\mathcal{M}_{\mathcal{D}})$ is equivalent to the domain wall topological phases $\mathbf{Fun}_{e|\mathcal{D}}(\mathcal{M}, \mathcal{M})$.
- The topological excitations are representations of tube algebra, thus tube algebra is charge symmetry.
- By boundary-bulk duality, the boundary tube algebra is the gauge symmetry of the bulk phase.

Example: Fibonacci tube algebra



Multimodule weak Hopf tube algebra

Suppose $\#I = k$, and $\#J = l$. For a multimodule category domain wall determined by a $\{\mathcal{C}_\alpha\}_{\alpha \in I} | \{\mathcal{D}_\beta\}_{\beta \in J}$ multimodule category \mathcal{M} , the domain wall tube algebra $\mathbf{Tube}(\{\mathcal{C}_\alpha\}_{\alpha \in I} | \{\mathcal{D}_\beta\}_{\beta \in J})$ is the vector space spanned by the following basis:

$$\left\{ \begin{array}{c} \begin{array}{c} v_{l+1} \\ \zeta_l \\ \zeta_1 \\ \gamma_k \\ \gamma_1 \\ u_1 \\ z_1 \\ \nu_1 \\ \nu_k \\ \mu_1 \\ \mu_l \\ w_{l+1} \end{array} \\ \begin{array}{c} a_k \cdots a_1 \quad b_1 \cdots b_l \end{array} \end{array} : \begin{array}{l} a_i \in \text{Irr}(\mathcal{C}_i), b_j \in \text{Irr}(\mathcal{D}_j), \\ u_i, z_i, v_j, w_j \in \text{Irr}(\mathcal{M}), \\ \text{wall vertex label} \in \text{Hom}_{\mathcal{M}}. \end{array} \right\}.$$

This is also a C^* weak Hopf algebra.

Boundary tube algebra

$$1 = \sum_{x, z \in \text{Irr}(\mathcal{M})} \left(\text{Diagram with dashed red arc} \right), \quad \mu \left(\left(\text{Diagram with arc } a \right) \otimes \left(\text{Diagram with arc } a' \right) \right) = \delta_{u, v'} \delta_{z, y'} \left(\text{Diagram with arcs } a, a' \right),$$

$$\varepsilon \left(\left(\text{Diagram with arc } a \right) \right) = \frac{\delta_{z, u} \delta_{y, v}}{d_y^L} \bar{y} \left(\text{Diagram with arc } a \right), \quad \Delta \left(\left(\text{Diagram with arc } a \right) \right) = \sum_{i, j, \sigma} \sqrt{\frac{d_j^L}{d_a d_i^L}} \left(\text{Diagram with arcs } a, i \right) \otimes \left(\text{Diagram with arcs } j, \sigma \right),$$

$$S \left(\left(\text{Diagram with arc } a \right) \right) = \frac{d_u^L}{d_v^L} \bar{a} \left(\text{Diagram with arc } a \right), \quad \left(\text{Diagram with arc } a \right)^* = \frac{d_z^L}{d_y^L} \bar{a} \left(\text{Diagram with arc } a \right).$$

From Drinfeld quantum double to quantum N-tuple

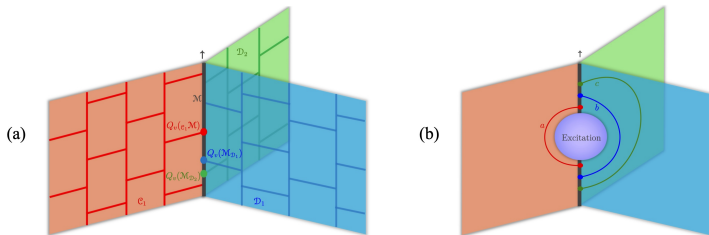
Bulk tube algebra as crossed product of boundary tube algebra:

$$\begin{aligned}
 \Phi : & \left(\text{Diagram 1} \otimes \text{Diagram 2} \right) \mapsto \mu \left(\text{Diagram 3} \otimes \text{Diagram 4} \right) \\
 & = \delta_{v,v'} \delta_{y,y'} \cdot \left(\text{Diagram 5} \right) .
 \end{aligned}$$

The diagrammatic equation illustrates the mapping Φ from the bulk tube algebra to the boundary tube algebra. On the left, the tensor product of two annuli is shown. The first annulus has a red dotted path γ and a blue solid path b , with labels w, v, y, μ, x and a central hole labeled 1 . The second annulus has a red solid path a and a blue dotted path ζ , with labels v', u, z, ν, y' and a central hole labeled 1 . This is mapped via μ to the tensor product of two annuli where the paths are swapped: the first has γ and b with labels w, v, y, μ, x , and the second has ζ and a with labels v', u, z, ν, y' . The result is then shown to be equal to $\delta_{v,v'} \delta_{y,y'}$ times a single annulus diagram with paths γ, ζ, a, b and labels $w, v, u, z, y, \nu, \mu, x$.

This is Drinfeld quantum double.

From Drinfeld quantum double to quantum N-tuple



If we glue N boundary tubes we obtain a multimodule domain wall tube, this extends the notion of Drinfeld quantum double to **quantum N-tuple algebra**.

Morita equivalence of tube algebra

General tube space $\mathbf{T}^{m_0, m_1; n_0, n_1}$ spanned by the tube string-net configurations :

$$\mathbf{T}^{m_0, m_1; n_0, n_1} = \text{span} \left\{ \begin{array}{c} \text{Diagram} \\ \text{ : edge} \in \text{Irr, vertex} \in \text{Hom} \end{array} \right\}$$

Morita equivalence

The tube space $\mathbf{T}^{m, m; n, n}$ are algebras for all $m, n \in \mathbb{N}$. The tube space $\mathbf{T}^{m, s; n, t}$ forms a right- $\mathbf{T}^{m, m; n, n}$ and left- $\mathbf{T}^{s, s; t, t}$ bimodule. These structures form a Morita context in the sense that

$$\mathbf{T}^{m, s; n, t} \otimes_{\mathbf{T}^{m, m; n, n}} \mathbf{T}^{s, m; t, n} \cong \mathbf{T}^{s, s; t, t}, \quad \mathbf{T}^{s, m; t, n} \otimes_{\mathbf{T}^{s, s; t, t}} \mathbf{T}^{m, s; n, t} \cong \mathbf{T}^{m, m; n, n}.$$

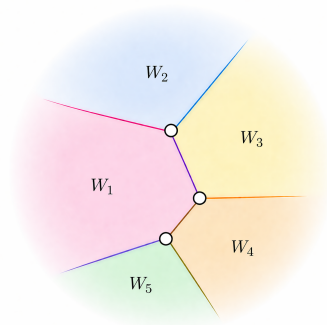
Thus $\mathbf{T}^{m, m; n, n}$'s are Morita equivalent for all $m, n \in \mathbb{N}$.

Weak Hopf symmetry behind $2d$ non-chiral topological order

- There are two type of weak Hopf symmetries for non-chiral topological phase: weak Hopf gauge symmetry and weak Hopf charge symmetry
- The weak Hopf sym is not unique, they are related by categorical Morita equivalence.
- For (weak Hopf) quantum double model, the weak Hopf gauge symmetry is the input weak Hopf algebra W , the weak Hopf charge symmetry is its quantum double $D(W)$.
- For (multifusion) string-net model, the weak Hopf gauge symmetry is given by boundary tube algebra, the weak Hopf charge symmetry is given by the bulk tube algebra.

Summary about WHA tube algebra

	Tube algebra description	Categorical description
Boundary phase	$\mathbf{Tube}(e\hat{\mathcal{M}})$	$\text{Boundary}(e\hat{\mathcal{M}}) \simeq \text{Rep}(\mathbf{Tube}(e\hat{\mathcal{M}}))$
Boundary defect	$\mathbf{Tube}(e\hat{\mathcal{M}}; e\hat{\mathcal{N}})$	$\text{Defect}(e\hat{\mathcal{M}}; e\hat{\mathcal{N}}) \simeq \text{Rep}(\mathbf{Tube}(e\hat{\mathcal{M}}; e\hat{\mathcal{N}}))$
Domain wall phase	$\mathbf{Tube}(e\hat{\mathcal{M}}_{\mathcal{D}})$	$\text{Wall}(e\hat{\mathcal{M}}_{\mathcal{D}}) \simeq \text{Rep}(\mathbf{Tube}(e\hat{\mathcal{M}}_{\mathcal{D}}))$
Domain wall defect	$\mathbf{Tube}(e\hat{\mathcal{M}}_{\mathcal{D}}; e\hat{\mathcal{N}}_{\mathcal{D}})$	$\text{Defect}(e\hat{\mathcal{M}}_{\mathcal{D}}; e\hat{\mathcal{N}}_{\mathcal{D}}) \simeq \text{Rep}(\mathbf{Tube}(e\hat{\mathcal{M}}_{\mathcal{D}}; e\hat{\mathcal{N}}_{\mathcal{D}}))$
Multimodule domain wall phase	$\mathbf{Tube}(\{e_{\alpha}\}\hat{\mathcal{M}}_{\{\mathcal{D}_{\beta}\}})$	$\text{Wall}(\{e_{\alpha}\}\hat{\mathcal{M}}_{\{\mathcal{D}_{\beta}\}}) \simeq \text{Rep}(\mathbf{Tube}(\{e_{\alpha}\}\hat{\mathcal{M}}_{\{\mathcal{D}_{\beta}\}}))$
Multimodule domain wall defect	$\mathbf{Tube}(\{e_{\alpha}\}\hat{\mathcal{M}}_{\{\mathcal{D}_{\beta}\}}; \{e_{\alpha}\}\hat{\mathcal{N}}_{\{\mathcal{D}_{\beta}\}})$	$\text{Defect}(\{e_{\alpha}\}\hat{\mathcal{M}}_{\{\mathcal{D}_{\beta}\}}; \{e_{\alpha}\}\hat{\mathcal{N}}_{\{\mathcal{D}_{\beta}\}})$ $\simeq \text{Rep}(\mathbf{Tube}(\{e_{\alpha}\}\hat{\mathcal{M}}_{\{\mathcal{D}_{\beta}\}}; \{e_{\alpha}\}\hat{\mathcal{N}}_{\{\mathcal{D}_{\beta}\}}))$

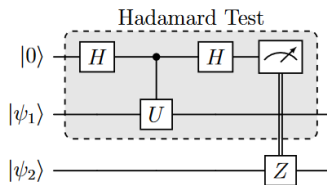


Weak Hopf cluster state and lattice SymTFT

Cluster State and MBQC

Quantum Circuit Model:

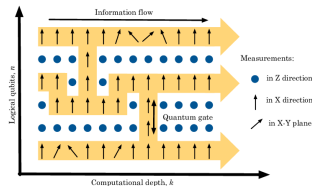
- Input: Product state $|0\rangle^{\otimes n}$
- Evolution: Sequential unitary gates $U_f = U_{t_n} \cdots U_{t_1}$
- Output: Measurement of final state



→ No specific entanglement structure required

Measurement-Based Quantum Computation:

- Input: Entangled multipartite state (**graph/cluster state**)
- Evolution: Sequential quantum measurements
- Output: Computational result from measurement outcomes



→ Specific entanglement pattern essential

Qubit cluster states

Definition

Given a graph $G = (V, E)$, the corresponding **graph/cluster state** is constructed as:

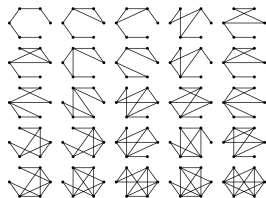
$$|G\rangle = \prod_{(i,j) \in E} CZ_{ij} |+\rangle^{\otimes n}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and CZ_{ij} applies a controlled-Z gate between vertices i and j .

Stabilizer Formulation: For each vertex $i \in V$, define the stabilizer generator:

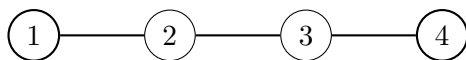
$$K_i = X_i \prod_{j \in N(i)} Z_j$$

where $N(i)$ denotes the neighbors of vertex i . The cluster state satisfies: $K_i |G\rangle = |G\rangle \quad \forall i \in V$



- ① **Hamiltonian formulation:** Since all stabilizer generators commute ($[K_i, K_j] = 0$ for all i, j), we can define a gapped Hamiltonian $H = -\sum_i K_i$. The cluster state is the unique ground state.
- ② **Symmetry-protected topological (SPT) phase:** Cluster states are short-range entangled with protected edge modes at boundaries.
- ③ **Computational resource:** SPT phases provide a natural framework for measurement-based quantum computation (MBQC).
- ④ **1d SPT:** The 1d cluster state model realizes a $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phase protected by on-site symmetries.

Example: 1D Cluster State with Open Boundaries

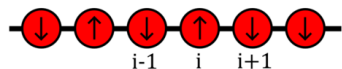
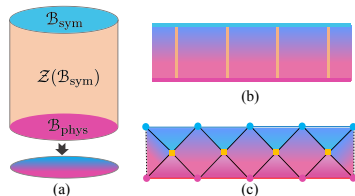


Cluster ladder model realizes SymTFT in (1+1)D

Lattice Model for (1+1)D SymTFT

The SymTFT can be viewed as a weak Hopf lattice gauge theory (equivalently, a string-net model) defined on a ladder with two types of boundaries:

- **Symmetry boundary and physical boundary.**
- Smooth + rough boundaries \Rightarrow cluster state model.
- General topological boundary + physical boundary \Rightarrow cluster ladder model.



$$H = - \sum_v A_v - \sum_f B_f$$

$$A_v = X \otimes X \otimes X$$

$$B_f = Z \otimes Z \otimes Z$$

Hopf algebra extension

Irrep basis	$ \Gamma_{ij}\rangle = \sqrt{d_\Gamma \mathcal{A} } \sum_{(\lambda)} \Gamma_{ij}(\lambda^{(1)}) \lambda^{(2)}\rangle$
Regular action	$\vec{X}_g h\rangle = gh\rangle, \overleftarrow{X}_g h\rangle = hS(g)\rangle$ $\vec{X}_g h\rangle = S(g)h\rangle, \overleftarrow{X}_g h\rangle = hg\rangle$
Irrep action	$Z_\Gamma h\rangle = \sum_{(h)} h^{(1)}\rangle \otimes \Gamma(h^{(2)}) \quad Z_\Gamma^\dagger h\rangle = \sum_{(h)} \Gamma(S(h^{(1)})) \otimes h^{(2)}\rangle$ $\tilde{Z}_\Gamma = \sum_{(h)} \Gamma(h^{(1)}) \otimes h^{(2)}\rangle \quad \tilde{Z}_\Gamma^\dagger = \sum_{(h)} h^{(1)}\rangle \otimes \Gamma(S(h^{(2)}))$
Symmetry action	$J_\Gamma = \text{Tr}' Z_\Gamma, J_\Gamma^\dagger = \text{Tr}' Z_\Gamma^\dagger$ $\tilde{J}_\Gamma = \text{Tr}' \tilde{Z}_\Gamma, \tilde{J}_\Gamma^\dagger = \text{Tr}' \tilde{Z}_\Gamma^\dagger$
Generalized Pauli operators	$X = \vec{X}_\lambda = \overleftarrow{X}_\lambda = \vec{X}_\lambda = \overleftarrow{X}_\lambda = \tilde{X}$ $Z = \sum_{\Gamma \in \text{Irr}(\mathcal{A})} \frac{d_\Gamma}{ \mathcal{A} } J_\Gamma = \sum_{\Gamma \in \text{Irr}(\mathcal{A})} \frac{d_\Gamma}{ \mathcal{A} } J_\Gamma^\dagger$ $\tilde{Z} = \sum_{\Gamma \in \text{Irr}(\mathcal{A})} \frac{d_\Gamma}{ \mathcal{A} } \tilde{J}_\Gamma = \sum_{\Gamma \in \text{Irr}(\mathcal{A})} \frac{d_\Gamma}{ \mathcal{A} } \tilde{J}_\Gamma^\dagger$ $Z = \tilde{Z}$

Hopf algebra extension

Total space	$\mathcal{H}_{tot} = \otimes_{v \in V} \mathcal{H}_v, \mathcal{H}_v = \mathcal{A}$
Z-preferred state	$ 1_{\mathcal{A}}\rangle, Z 1_{\mathcal{A}}\rangle = 1_{\mathcal{A}}\rangle$
X-preferred state	$ 1\rangle = \sqrt{ \mathcal{A} } \lambda\rangle, X 1\rangle = 1\rangle$
Preferred product state	$ \Omega\rangle = (\otimes_{v_o} \lambda\rangle_{v_o}) \otimes (\otimes_{v_e} 1_{\mathcal{A}}\rangle_{v_e})$
Edge entangler	$C\vec{X}_{i,j} : g\rangle_i h\rangle_j \mapsto \sum_{(g)} g^{(2)}\rangle_i g^{(1)}h\rangle_j$ $C\vec{X}_{i,j}^{\rightarrow -1} : g\rangle_i h\rangle_j \mapsto \sum_{(g)} g^{(2)}\rangle_i S(g^{(1)})h\rangle_j$ $C\overleftarrow{X}_{i,j} : g\rangle_i h\rangle_j \mapsto \sum_{(g)} g^{(2)}\rangle_i hS(g^{(1)})\rangle_j$ $C\overleftarrow{X}_{i,j}^{-1} : g\rangle_i h\rangle_j \mapsto \sum_{(g)} g^{(2)}\rangle_i hg^{(1)}\rangle_j$

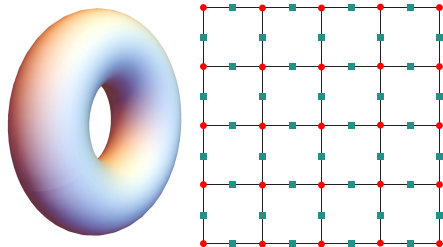
Generalized cluster states

Definition (Hopf cluster-lattice state)

For a cluster lattice and a Hopf algebra \mathcal{A} , the Hopf cluster state is defined as

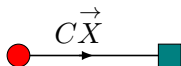
$$|\mathbb{M}^d, \mathcal{A}\rangle = \left(\prod_{j=1}^{|E|} U_{e_j} \right) |\Omega\rangle.$$

where $|\Omega\rangle = (\otimes_{v_o} |\mathbf{1}\rangle_{v_o}) \otimes (\otimes_{v_e} |1_{\mathcal{A}}\rangle_{v_e})$.

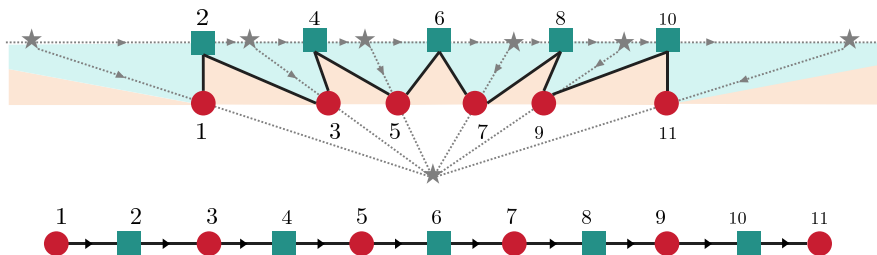
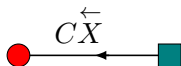


Generalized cluster states

- If the edge is directed from an odd vertex to an even vertex, we set $U_e = C\vec{X}$:



- If the edge is directed from an even vertex to an odd vertex, we set $U_e = C\overleftarrow{X}$:



Hopf cluster state Hamiltonian

- For odd vertex i , we define X-type local stabilizer operator as

$$\mathfrak{A}_i = \sum_{(\lambda)} \overleftarrow{X}_{\lambda(1)}(i-1) \otimes \overrightarrow{X}_{\lambda(3)}(i) \otimes \overrightarrow{X}_{\lambda(2)}(i+1).$$

We can similarly define \mathfrak{A}^h for $h \in \mathcal{A}$.

- For even vertex j , we define the Z-type operator

$$\mathfrak{B}_j^\Gamma = \text{Tr}'[Z_\Gamma^\dagger(j-1) \otimes Z_\Gamma(j) \otimes Z_\Gamma(j+1)]$$

and

$$\mathfrak{B}_j = \sum_{\Gamma \in \text{Irr}(\mathcal{A})} \frac{d_\Gamma}{|\mathcal{A}|} \mathfrak{B}_j^\Gamma.$$

- The Hopf cluster state lattice Hamiltonian

$$H_{\text{cluster}} = - \sum_{i:\text{odd}} \mathfrak{A}_i - \sum_{j:\text{even}} \mathfrak{B}_j.$$

- The Hopf cluster state $|\mathbb{M}^1, \mathcal{A}\rangle$ is the ground state.
- Symmetry $\text{Rep}(\mathcal{A}) \times \text{Rep}(\mathcal{A}^\vee) = \text{Rep}(\mathcal{A}) \times \text{Cocom}(\mathcal{A})$.

- The non-invertible SPT and MBQC.
- Higher dimensional model and higher category structure. Higher dim lattice SymTFT.
- Entanglement property, entangle entropy is sensitive to defect and boundary.
- SET/SPT generalization of quantum double model and string-net model.
- Anyon condensation theory from weak Hopf symmetry breaking.
- Operator algebra perspective: stability, Haag duality, infinite-volume sector, etc.